Categorification of 1 and of the Alexander polynomial

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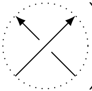
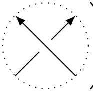
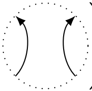
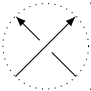
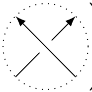
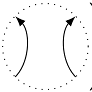
QUACKS

- The $\mathfrak{gl}_1$ link invariant P_1.

  $qP_1\Bigg($  $\Bigg) - q^{-1}P_1\Bigg($  $\Bigg) = (q-q^{-1})P_1\Bigg($  $\Bigg)$

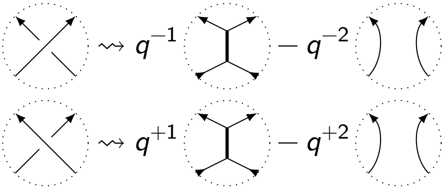
  $L \mapsto 1 \in \mathbb{Z}[q, q^{-1}]$

- The Alexander polynomial Δ.

  $\Delta\Bigg($  $\Bigg) - \Delta\Bigg($  $\Bigg) = (q-q^{-1})\Delta\Bigg($  $\Bigg)$

$\mathfrak{gl}_1$ invariant

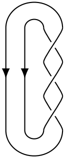
Link diagram $\rightsquigarrow$ $\mathbb{Z}[q, q^{-1}]$-lin. comb. of plane graphs



plane graph $\rightsquigarrow$ element of $\mathbb{N}[q, q^{-1}]$

$$\Gamma \rightsquigarrow \left(q + q^{-1}\right)^{\#V(\Gamma)/2} = [2]^{\#V(\Gamma)/2}.$$

$\mathfrak{gl}_1$ invariant – Example



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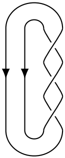
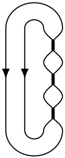


q^{-3} 

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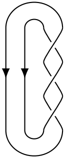

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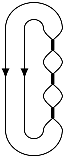
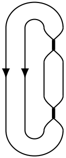
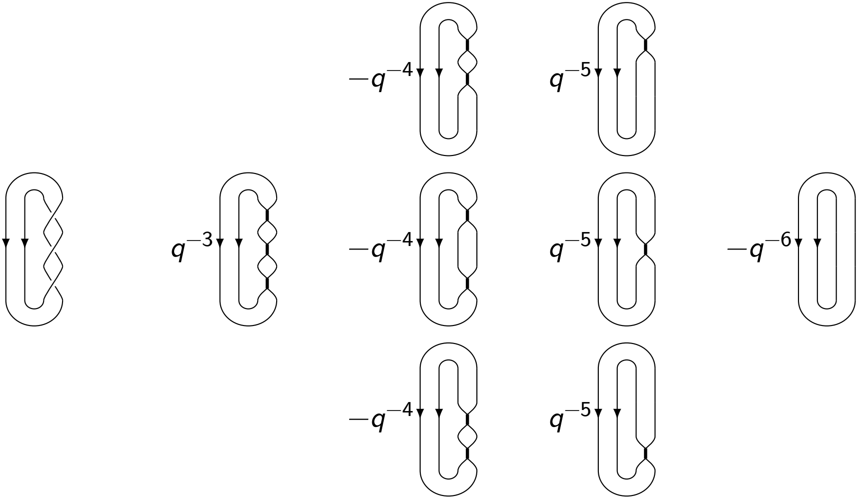


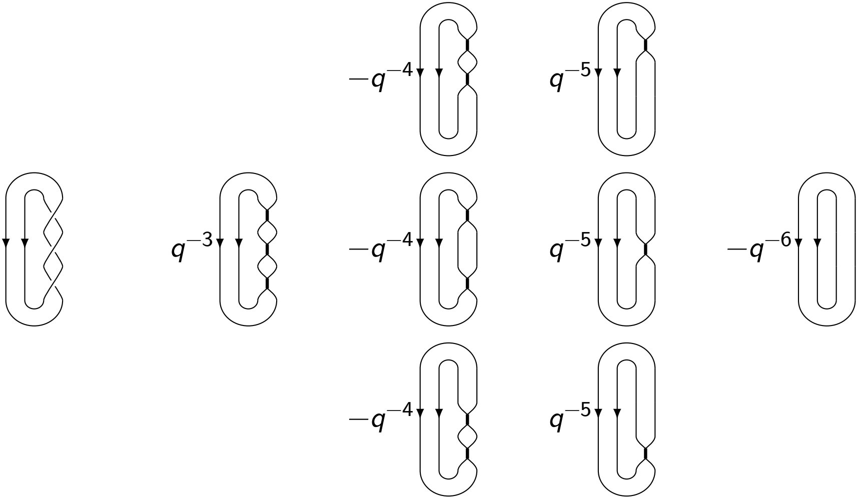
$-q^{-4}$



$\mathfrak{gl}_1$ invariant – Example



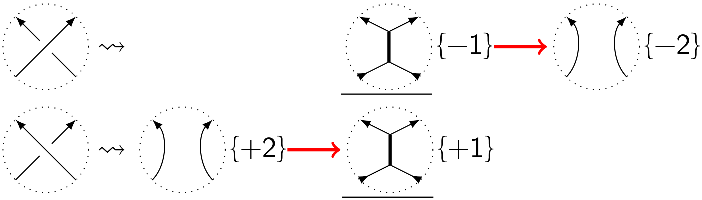
$\mathfrak{gl}_1$ invariant – Example



$$1 = q^{-3}[2]^3 \quad -3q^{-4}[2]^2 \quad +3q^{-5}[2] \quad -q^{-6}$$

$\mathfrak{gl}_1$-homology

Braid closure diagram $\rightsquigarrow$ hypercube of plane graphs graphs (with shifts)



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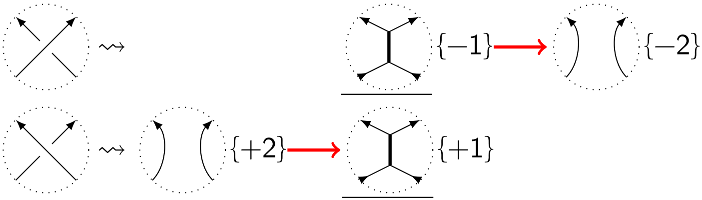
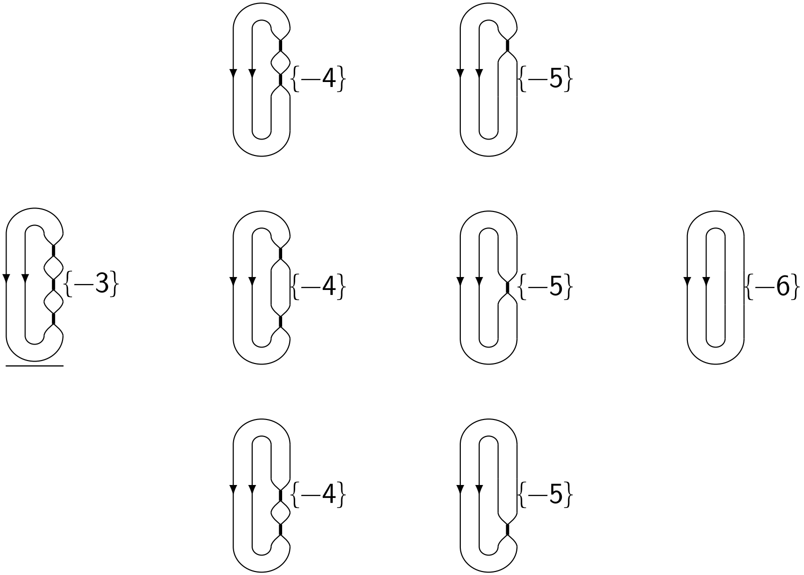


Planar (vinyl) graph $\rightsquigarrow$graded vector space

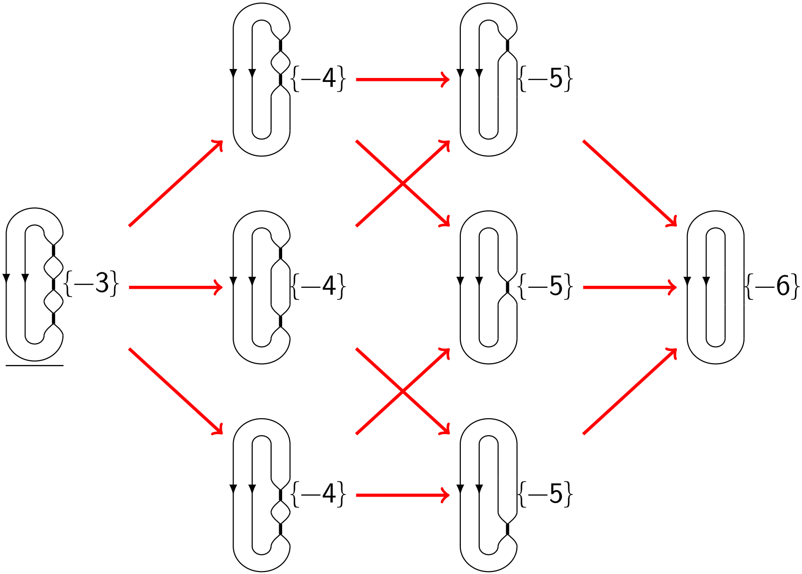
dimension $[2]^{\#V(\Gamma)/2}$

$\longrightarrow$ $\rightsquigarrow$graded linear map

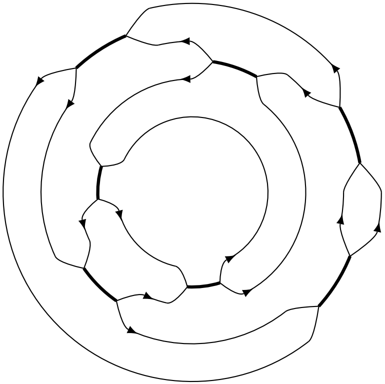
$\mathfrak{gl}_1$ homology – Example



$\mathfrak{gl}_1$ homology – Example

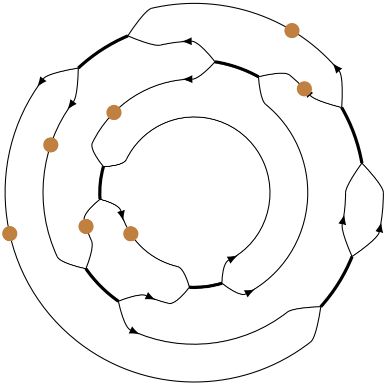


Vinyl graph ⇝ vector space



Vinyl graph $\Gamma \circlearrowleft$ index k.

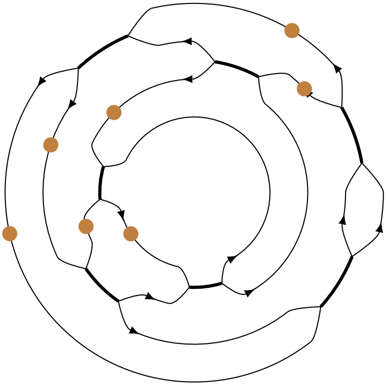
Vinyl graph $\rightsquigarrow$ vector space



Vinyl graph Γ $\circlearrowleft$ index k.

Dot configuration d ●.

Vinyl graph $\rightsquigarrow$ vector space



Vinyl graph Γ ⟲ index k.
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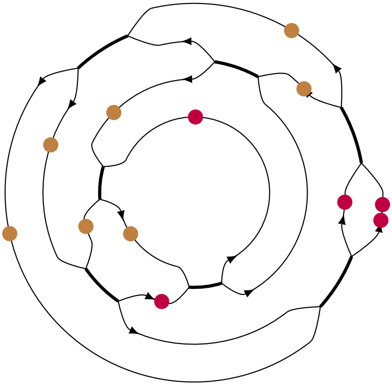
$$D(\Gamma) = \bigoplus_{d} \mathbb{Q}.$$

Vinyl graph $\rightsquigarrow$ vector space



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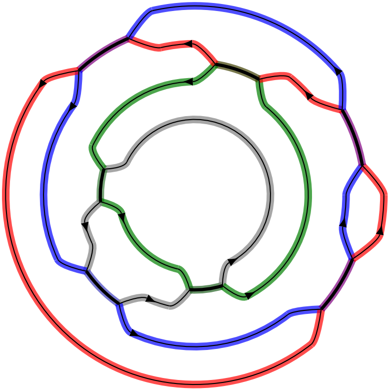
Multiplication μ on $D(\Gamma)$.

Vinyl graph $\rightsquigarrow$ vector space



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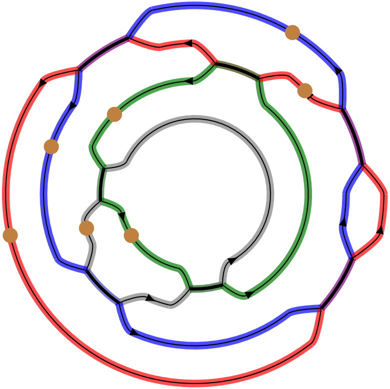
$$D(\Gamma) = \bigoplus_{d} \mathbb{Q}.$$

Multiplication μ on $D(\Gamma)$.
Coloring $c = (C_1, C_2, \ldots, C_k)$
$\Gamma = \bigsqcup_i C_i$.

Vinyl graph ⇝ vector space



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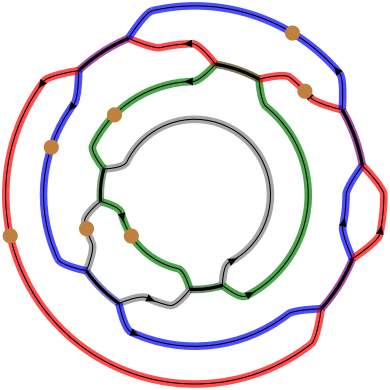
$$\tau(d,c) = \frac{\prod_{i=1}^{k} X_i^{\#\{\bullet \text{ in } C_i\}}}{\prod_{C_i\ C_j} (X_i - X_j)}$$

Vinyl graph $\rightsquigarrow$ vector space



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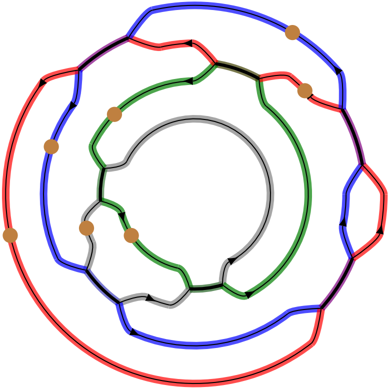
$$\tau(d,c) = \frac{\prod_{i=1}^{k} X_i^{\#\{\bullet \text{ in } C_i\}}}{\prod (X_i - X_j)} = \frac{-X_1{}^2 X_2{}^2 X_3 X_4{}^2}{(X_1 - X_2)^3 (X_3 - X_4)^2 (X_1 - X_4)(X_2 - X_3)}$$

C_i C_j

Vinyl graph $\rightsquigarrow$ vector space



Vinyl graph Γ $\circlearrowleft$ index k.
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$$\tau(d,c) = \frac{\prod_{i=1}^{k} X_i^{\#\{\bullet \text{ in } C_i\}}}{\prod (X_i - X_j)}$$

$$\tau(d) = \sum_{c \in \mathrm{col}(\Gamma)} \tau(d,c)$$

Vinyl graph $\rightsquigarrow$ vector space

Proposition (Robert–W., '17)

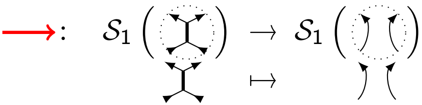
For any dot configuration d, $\tau(d) \in \mathbb{Q}[X_1, \dots, X_k]^{S_k}$.

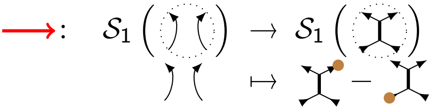
$$\mathcal{S}_1(\Gamma) = D(\Gamma)/\ker(\tau \circ \mu(\_, \_)_{X_\bullet \mapsto 0}).$$

Theorem (Robert–W., '18)

For any vinyl graph Γ, $\dim_q \mathcal{S}_1(\Gamma) = [2]^{\# V(\Gamma)/2}$.

$\longrightarrow$ $\rightsquigarrow$ linear map





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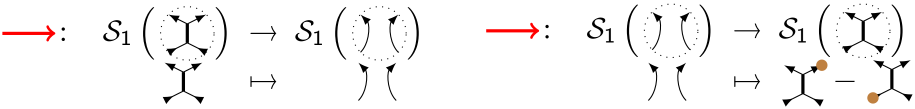


Theorem (Robert.–W. '18)

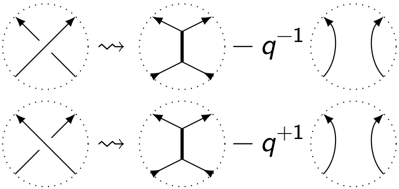
1. *These maps in the flattening of the hypercube produces a chain complex. Its homology, denoted $H_{\mathfrak{gl}_1}$ is a link invariant which categorifies P_1.*
2. *There is a spectral sequence from the triply graded homology to $H_{\mathfrak{gl}_1}$.*

Examples

1. Trefoil: the Poincaré polynomial is $1 + q^{-4}(t + t^2)$.
2. Hopf link: the Poincaré polynomial is $1 + q^2(1 + t)$.

Alexander polynomial

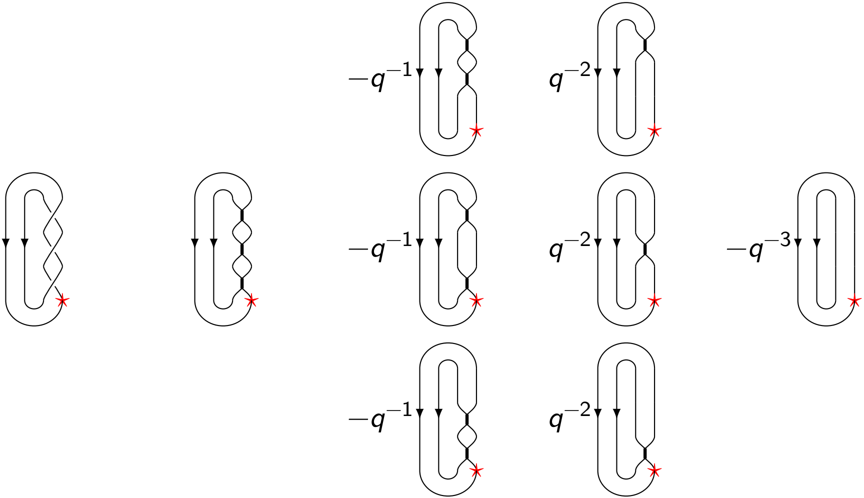
Marked ($\star$) braid closure $\rightsquigarrow \mathbb{Z}[q, q^{-1}]$-lin. comb. of marked plane graphs



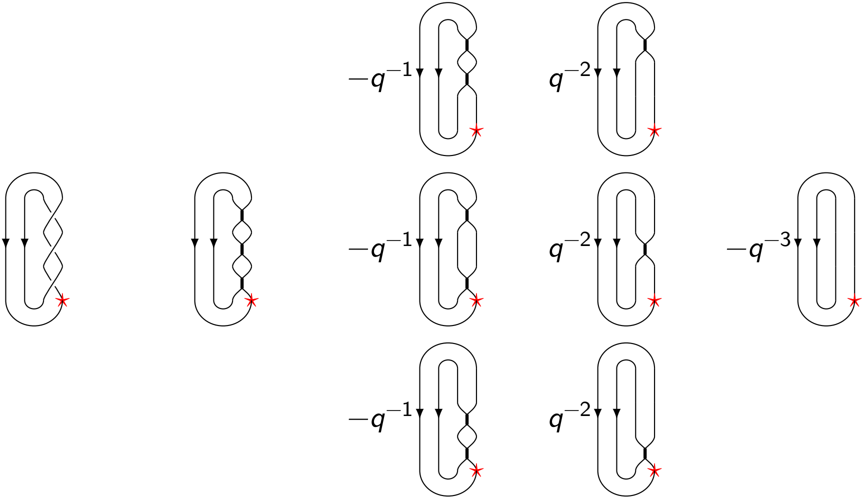
Marked plane graph $\rightsquigarrow$ element of $\mathbb{N}[q, q^{-1}]$

$\Gamma \rightsquigarrow$ complicated (comes from $U_q(\mathfrak{gl}(1|1)) - \text{mod}$).

Alexander polynomial – Example



Alexander polynomial – Example



$$q^2 - 1 + q^{-2} \; = \quad [2]^2 \quad -3q^{-1}[2] \quad +3q^{-2} \quad 0$$

$\mathfrak{gl}_0$homology

Same hypercube with a different functor.

$\mathfrak{gl}_0$ homology

Same hypercube with a different functor.

$$\mathcal{S}_0'(\Gamma_\star) \subseteq \mathcal{S}_1(\Gamma) = \langle \text{at least } k-1 \bullet \text{ at } \star \rangle\{-k+1\}$$

$\longrightarrow$ $\rightsquigarrow$ induced by $\mathcal{S}_1$.

$\mathfrak{gl}_0$ homology

Same hypercube with a different functor.

$$\mathcal{S}_0'(\Gamma_\star) \subseteq \mathcal{S}_1(\Gamma) = \langle \text{at least } k-1 \bullet \text{ at } \star \rangle\{-k+1\}$$

$\longrightarrow \;\rightsquigarrow$ induced by $\mathcal{S}_1$.

Theorem (Robert–W., '19)

For any right-marked vinyl graph $\Gamma_\star$, $\dim_q \mathcal{S}_0'(\Gamma_\star)$ is the expected graded dimension.

Theorem (Robert–W. '19)

1. *The flattening of the hypercube with $\mathcal{S}_0'$ produces a chain complex. Its homology, denoted $H_{\mathfrak{gl}_0}$ is a knot invariant which categorifies the Alexander polynomial.*
2. *There is a spectral sequence from the reduced triply graded homology to $H_{\mathfrak{gl}_0}$.*