Howe to translate Gelfand-Tsetlin

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Definition

The category $\mathscr{S}_c$ of **Soergel bimodules** *(for $\mathfrak{gl}_n$) is the Karoubi envelope of the monoidal subcategory of graded bimodules over $S = \mathbb{C}[x_1, \dots, x_n]$ generated by the bimodules $B_i = S \otimes_{S^{s_i}} S$.*

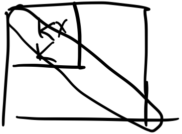
Different people appreciate this category because of the multifarious ways its appears in mathematics.

1. combinatorially either as above, or in terms of Soergel calculus (which I won't describe in detail here).
2. representation theoretically in terms of translation and projective functors.
3. geometrically in terms of perverse sheaves on flag varieties.

Let me give a quick sketch of these constructions.

Basic notation for $\mathfrak{g} = \mathfrak{gl}_n$:

- A weight is given by an n-tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. Dominant integral if $\lambda_i \in \mathbb{Z}$, and $\lambda_1 \leq \cdots \leq \lambda_n$.
- The center $Z_n = Z(U(\mathfrak{g}))$ is isomorphic to $\mathbb{C}[z_1, \dots, z_n]^{S_n}$ with $f(z_1, \dots, z_n)$ acting by $f(\lambda_1 + 1, \dots, \lambda_n + n)$ on the Verma $M(\lambda)$.
- Let $I_0 = \{f \in Z \mid f(1, \dots, n) = 0\}$ be the annihilator of the trivial module, and $\widehat{Z}_n$ be the completion of Z_n at this ideal. This is isomorphic to the completion $\widehat{S} \cong \widehat{Z}_n$ via the map $x_i \mapsto z_i - i$.
- We have an inclusion $\iota_k \colon Z_k \hookrightarrow U(\mathfrak{gl}_k) \hookrightarrow U(\mathfrak{gl}_n)$. The subring Γ generated $\iota_k(Z_k)$ for $k = 1, \dots, n$ is called the **Gelfand-Tsetlin** subalgebra.



Definition

A module M over $U(\mathfrak{gl}_n)$ is **Gelfand-Tsetlin** *if it is Γ-locally finite, i.e. for any $m \in M$, we have* $\dim(\Gamma m) < \infty$.

Finite dimensional modules are obviously Gelfand-Tsetlin, as are Verma modules for all Borels containing torus (so all objects in categories $\mathcal{O}$).

Questions:

- What are the simple Gelfand-Tsetlin modules? (Hard; only solved in 2018 by KTWWY.)
- How does tensor product with finite dimensional modules act on GT modules? (Less hard, I'll explain today.)

For $\chi \in \mathrm{MaxSpec}(Z_n)$, let $\mathcal{C}_\chi$ be the subcategory of $U(\mathfrak{gl}_n)$ modules where a power of I_χ acts trivially, and $\mathrm{pr}_\chi : \mathfrak{g}\text{-mod} \to \mathcal{C}_\chi$ be functor of the largest subobject in this category.

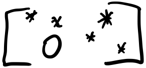
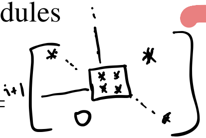
For any finite dimensional $\mathfrak{g}$-module U, we, have a functor $\mathrm{pr}_{\chi'}(U \otimes -) : \mathcal{C}_\chi \to \mathcal{C}_{\chi'}$. The category $\mathscr{S}_r(\chi, \chi')$ of **projective functors** are sums of summands of these.

Theorem (Bernstein-Gelfand, Soergel)

There's a tensor equivalence between $\mathscr{S}_r = \mathscr{S}_r(0,0)$ of projective functors $\mathcal{C}_0 \to \mathcal{C}_0$ and the category $\widehat{\mathscr{S}_c}$ of completed (ungraded) Soergel $\widehat{Z}_n$-$\widehat{Z}_n$-bimodules.

The bimodule ${}_{S^{s_i}}S_S$ corresponds to translation onto a wall with $x_i = x_{i+1}$, and ${}_SS_{S^{s_i}}$ to translation off.

On the other hand, we can also interpret these bimodules geometrically. Let

$$G = GL_n \qquad B = $$  $$P_i = $$ 

Soergel bimodules are an algebraic reflection of the geometry of the double coset space $B\backslash G/B$. This space has a category $\mathscr{S}_g$ of sum of shifts of semi-simple perverse sheaves inside the derived category, which is monoidal under convolution.

Theorem (Soergel)

The pushforward $B\backslash G/B \to B\backslash * /B$ induces a monoidal equivalence $\mathscr{S}_g \to \mathscr{S}_c$, matching homological grading of perverse sheaves to internal grading of Soergel bimodules, sending $IC(P_i)$ to $S \otimes_{S^{s_i}} S$.

These results together are the key to the self-Koszul duality of category $\mathscr{O}$ (since simple perverse sheaves correspond to projective functors).

So we have a sequence of functors of additive categories:

$$\mathscr{S}_g \to \mathscr{S}_c \to \mathscr{S}_r$$

graded lift of $\mathscr{S}_r$

with the first an equivalence, and the second an equivalence after completion and forgetting gradings.

We can extend this to the singular case as well. For each $\chi \in \mathrm{MaxSpec}_{\mathbb{Z}}(Z_n)$, we have a parabolic P_χ, corresponding invariant ring S^{W_χ}, and have equivalences:

$$\mathscr{S}_g(\chi, \chi') = \mathsf{Perv}(P_\chi \backslash G / P_{\chi'}) \to \mathscr{S}_c(\chi, \chi') \to \mathscr{S}_r(\chi, \chi')$$

$\otimes \mathbb{C}^n$

$\otimes (\mathbb{C}^n)^*$

The resulting 2-category is a quotient of categorified $\mathfrak{sl}_\infty$.

$S^{W_\chi} = H^*_{P_\chi}(*)$

Let's compare this with Joel's talk. Let $\mathcal{O}_\chi \subset \mathcal{C}_\chi$ is the category of weight modules which are $U(\mathfrak{b})$ locally finite.

Theorem (Joel's talk)

We have an isomorphism

controls χ

$$\bigoplus_{\chi\in \mathrm{MaxSpec}_{\mathbb{Z}}(\Gamma)} K^0(\mathcal{O}_\chi) \cong \mathrm{Sym}^n(\mathbb{C}^\infty \otimes \mathbb{C}^n)_0$$

$$K^0(GT_\chi) \cong U(\mathfrak{sl}_n^+) \otimes \mathrm{Sym}^n(\mathbb{C}^\infty \otimes \mathbb{C}^n)_0$$

to the 0-weight space for the $\mathfrak{sl}_n$-action on the RHS.

Joel's functors change the underlying algebra (to a finite W-algebra or worse). On the other hand, the projective functors give the commuting Howe dual action of $\mathfrak{sl}_\infty$. So, my talk is the Howe dual of Joel's.

$$U(\mathfrak{sl}_n^+) \otimes (\mathbb{C}^n)^{\otimes n}$$

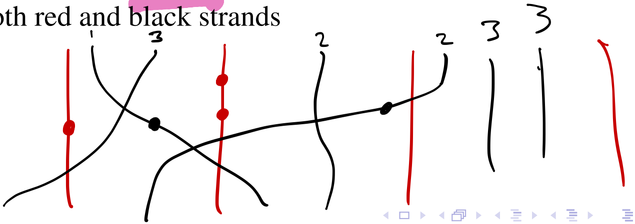
In [KTWWY], we didn't just identify category $\mathcal{O}$. We found the whole category of Gelfand-Tsetlin modules $\mathcal{GT}_\chi$.

Let $\tilde{\mathbb{T}}^\chi$ denote the KLRW algebra for the Dynkin diagram $1 - 2 - 3 - \cdots - (n-1)$, with

$z_i = a_i$

- red strands with x-values given by the entries of χ (when χ is singular, we get thick strands from the repeats), all labeled by the appropriate multiple of the fundamental weight ω_{n-1}.
- k black strands with label k for all $k = 1, \dots, n-1$.
- dots on both red and black strands

$n = 4$



Theorem

The category $\mathcal{GT}_\chi$ is equivalent to the category of weakly-graded finite dimensional $\tilde{\mathbb{T}}^\chi$-modules.

Under this equivalence, the images of the obvious idempotents in $\tilde{\mathbb{T}}^\chi$ match with the weight spaces for elements of MaxSpec(Γ).

- to get $\mathcal{O}_\chi$, kill idempotents corresponding to weight spaces not allowed in category $\mathcal{O}$. $1 - - - - - = 0$
- red dots = nilpotent part of Z_n action.
- sum of black dots on strands with label i = nilpotent part of $U(\mathfrak{h})$-action.

So, we have a Soergel bimodule action on $\tilde{\mathbb{T}}^0$-modules, and a categorical $\mathfrak{sl}_\infty$ action on all χ's together. How can we describe it in these combinatorial terms?

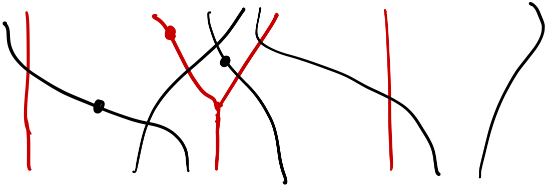
The talk thus far

$\mathcal{GT}_\chi \to \mathrm{Vect}$

geometry	combinatorics	representation theory
perverse sheaves on $P_\chi \backslash G / P_{\chi'}$	S^{W_χ} - $S^{W_{\chi'}}$ Soergel bimodules	projective functors $\mathcal{C}_\chi \to \mathcal{C}_{\chi'}$
??	??	usual action
??	$\tilde{\mathbb{T}}^\chi$-modules	Gelfand-Tsetlin modules $\mathcal{GT}_\chi$

Connection to KLSY

Translation onto/off of the wall corresponds to "splitter bimodules" from Khovanov-Lauda-Sussan-Yonezawa.



Key observation of the proof:

$$U(\mathfrak{gl}_n) \otimes \mathbb{C}^n \cong U(\mathfrak{gl}_n) E_n U(\mathfrak{gl}_n) \subset U(\mathfrak{gl}_{n+1})$$

Theorem (W.)

The monoidal category $\mathscr{S}_c$ acts on $\tilde{\mathbb{T}}^0$-modules via the KLSY bimodules.

Note, this shows why red-dotting was needed: projective functors don't preserve semi-simple action of the center.

Of course, this only covers very specific number of black strands, and in particular, only a few of the $\mathfrak{sl}_2$ cases KLSY consider.

One fix: generalize $U(\mathfrak{gl}_n)$ to other Coulomb branches.

- this includes finite W-algebras if $v_1 \leq v_2 - v_1 \leq \cdots \leq v_{n-1} - v_{n-2} \leq n - v_{n-1}$.
- other weirder stuff in other cases.

A bit tricky to write out details of, though.

Solution I prefer: get that last corner of my summary page, geometry.

Choose an m-tuple of integers $(v_1, \dots, v_{m-1}, v_m = n)$. Let

$$V = \operatorname{Hom}(\mathbb{C}^{v_1}, \mathbb{C}^{v_2}) \oplus \operatorname{Hom}(\mathbb{C}^{v_2}, \mathbb{C}^{v_3}) \oplus \cdots \oplus \operatorname{Hom}(\mathbb{C}^{v_{m-2}}, \mathbb{C}^{v_{m-1}}) \oplus \operatorname{Hom}(\mathbb{C}^{v_{m-1}}, \mathbb{C}^{n})$$

$$H_0 = GL(v_1) \times GL(v_2) \times \cdots \times GL(v_{m-1}) \qquad H = H_0 \times B$$

Almost a moduli of quiver reps, but note that B instead of a G.

$\circlearrowleft G$

Theorem (Guan-W.)

The H-orbits on V are classified by ways of writing $\mathbf{v}$ *as a sum of positive roots, with a choice of order on the roots of type* $(0, \dots, 0, 1, \dots, 1)$ *appearing.*

order

$$\begin{array}{ccccccc} \mathbb{C} & \to & \mathbb{C} & & 0 & \to & \mathbb{C} \\ & & 0 & \to & \mathbb{C} & \to & \mathbb{C} \\ & & \mathbb{C} & \to & \mathbb{C} & \to & \mathbb{C} \end{array}$$

Let $V_{\mathrm{inj}} \subset V$ be the subspace where all the maps $f_i \colon \mathbb{C}^{v_i} \to \mathbb{C}^{v_{i+1}}$ are injective.

In the case $\mathbf{v} = (1, 2, \ldots, n)$, we have a close relationship to the flag variety.

Lemma

We have a G-equivariant isomorphism $V_{\mathrm{inj}}/H_0 \cong B\backslash G$ by thinking of

$$\overset{GL_n}{\circlearrowright} \mathbb{C}^n \supset \overset{n-1}{\mathrm{im}(f_{n-1})} \supset \overset{n-2}{\mathrm{im}(f_{n-1}f_{n-2})} \supset \cdots \supset \overset{1}{\mathrm{im}(f_{n-1}\cdots f_1)}$$

as a flag.

So, we have a category of sums of shifts of semi-simple perverse sheaves $\mathsf{Perv}(V/H)$.

Theorem

The category of $\mathsf{Perv}(V/H)$ *carries an action by convolution of* $\mathscr{S}_g = \mathsf{Perv}(B\backslash G/B)$ *via convolution.*

This is a general observation about spaces with a G-action that we restrict to the action of B.

If we let $H_i = H_0 \times P_i$, then the action of $IC(P_i)$ is pushing and pulling on the map $V/H \to V/H_i$.

As usual, we can generalize to the singular case by considering $\mathsf{Perv}(V/H_\chi)$ for $H_\chi = H_0 \times P_\chi$.

n red strands

As you might expect, this matches with the other categories with Soergel actions:

v_i black strands label i

Theorem

The category Perv(V/H_χ) *is equivalent to the category of graded projective $\tilde{\mathbb{T}}^\chi$-modules. This intertwines the Soergel action on $\tilde{\mathbb{T}}^0$-modules and $\mathfrak{sl}_\infty$-action on all χ with that by KLSY bimodules.*

This is the easiest way to prove that such an action exists. Of course, you have to work algebraically if want to do p-DG (for now).

Restricting to V_{inj} has effect of passing to $\mathscr{O}$, back to Joel's talk.

Proof analogous to [VV]

The whole talk

geometry	combinatorics	representation theory
perverse sheaves on $P_\chi \backslash G / P_{\chi'}$	S^{W_χ} - $S^{W_{\chi'}}$ Soergel bimodules	projective functors $\mathcal{C}_\chi \to \mathcal{C}_{\chi'}$
convolution	KLSY bimodules	usual action
perverse sheaves on V/H_χ	$\tilde{\mathbb{T}}^\chi$-modules	Gelfand-Tsetlin modules $\mathcal{GT}_\chi$

Thanks for listening.