Invariants of 4-manifolds from Khovanov-Rozansky link homology

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Starting in dimension 3...

Link invariants

The $\mathfrak{gl}_N$ link polynomial $P_N\colon \{\text{framed, oriented links}\} \to \mathbb{Z}[q^{\pm 1}]$:

$$P_N(\nearrow\!\!\!\!\nwarrow) - P_N(\nwarrow\!\!\!\!\nearrow) = (q - q^{-1}) P_N(\uparrow\uparrow)$$

$$P_N(\nearrow\!\!\circlearrowleft) = q^N P_N(\uparrow), \quad P_N(L_1 \sqcup L_2) = P_N(L_1) P_N(L_2)$$

Higher categories

Ribbon category $\mathrm{Rep}(U_q(\mathfrak{gl}_N))$, RT tangle invariants

Manifold invariants

The $\mathfrak{gl}_N$ skein module for compact, oriented M^3, $P \subset \partial M^3$:

$$\mathrm{Sk}_N(M^3; P) := \frac{\mathbb{Z}[q^{\pm 1}]\langle \text{framed, oriented tangles in } (M^3, P)\rangle}{\langle \text{isotopy, local relations in } B^3 \hookrightarrow M^3\rangle}$$

Part of a 0123ε-dimensional TFT.

...upgrading to dimension 4

Khovanov–Rozansky 2004, Robert–Wagner+Ehrig–Tubbenhauer–W 2017:

Link invariants

The $\mathfrak{gl}_N$ Khovanov–Rozansky link homology

$$\mathrm{KhR}_N \colon \{\text{links/link cobordisms}\} \to K^b(\mathrm{gr}^{\mathbb{Z}}\mathrm{Vect}), \quad \chi_q \circ \mathrm{KhR}_N = P_N$$

Morrison–Walker–W 2019:

Higher categories

A ribbon 2-category / disk-like 4-category $\mathrm{CatRep}(U_q(\mathfrak{gl}_N))$.

Manifold invariants

A 'skein module' $\mathcal{S}_N(W^4; L)$ for compact, oriented, smooth W^4, $L \subset \partial W^4$.

$$\mathcal{S}_N(B^4; L) \cong \mathrm{KhR}_N(L).$$

Expected to be part of a 01234ε-dimensional TFT.

Approaches

Routes to Khovanov–Rozansky homology for (links in) 3-manifolds:

- Categorify Witten–Reshetikhin–Turaev invariants
  - Categorification of tensor product reps & at roots of unity
  - RepCat($U_q(\mathfrak{gl}_N)$)
- Categorify 3D skein modules
  - Via surgery
  - Via Heegaard splitting, categorified skein algebras
- Extending Witten's model for Khovanov homology in $\mathbb{R}^3$
- **Higher skein modules** (this talk)
  - Functorial tangle invariant $\to$ 4-category $\to$ 4D skein module

Khovanov–Rozansky homology

$$
\begin{array}{ccc}
\left\{ \begin{array}{c} \text{links diagrams} \\ \text{movies of diagrams/m. moves} \end{array} \right\} & \xrightarrow{\ \mathrm{KhR}_N\ } & K^b(\mathrm{gr}^{\mathbb{Z}}\mathrm{Vect}) \\
\updownarrow \cong & & \downarrow \chi_q \\
\left\{ \begin{array}{c} \text{links embedded in } B^3 \\ \text{cobordisms in } B^3 \times I/\text{isotopy} \end{array} \right\} & \xrightarrow{\ P_N \circ K_0\ } & \mathbb{Z}[q^{\pm 1}]
\end{array}
$$

Defining KhR_N requires:

- the data of a chain complex for each link diagram (KhR04)
- the data of a chain map for every elementary movie (KhR04)
- movie move checks (Blanchet10, ETW17)

Khovanov–Rozansky homology

$$\left\{ \begin{array}{c} \text{links diagrams} \\ \text{movies of diagrams/m. moves} \end{array} \right\} \xrightarrow{\mathrm{KhR}_N} K^b(\mathrm{gr}^{\mathbb{Z}}\mathrm{Vect})$$

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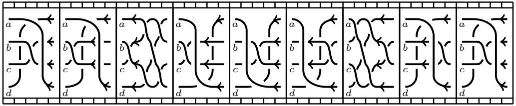
E.g. the composite of these R3 chain maps



MM10

should be homotopic to the identity.

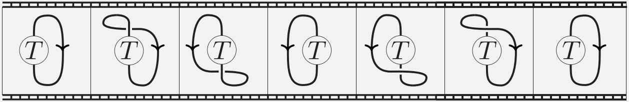
Defir

- the data of a chain complex for each link diagram (KhR04)
- the data of a chain map for every elementary movie (KhR04)
- movie move checks (Blanchet10, ETW17)

Functoriality in S^3

For $\mathcal{S}_N(B^4; L) \cong \mathrm{KhR}_N(L)$ we need KhR_N for links in $S^3 = B^3 \cup \{\infty\}$.

- links in S^3 generically avoid ∞
  $\Longrightarrow$ same chain complexes
- link cobordisms in $S^3 \times I$ generically avoid $\infty \times I$
  $\Longrightarrow$ same chain maps
- link cobordism isotopies in $S^3 \times I^2$ might intersect $\infty \times I^2$ transversely
  $\Longrightarrow$ a new movie move to check, non-local if viewed from B^3

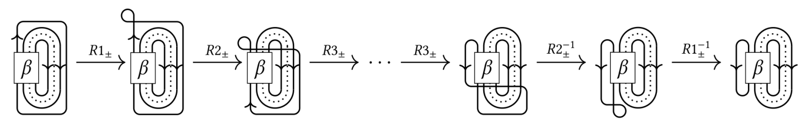


Theorem (M.-W.-W. 2019)

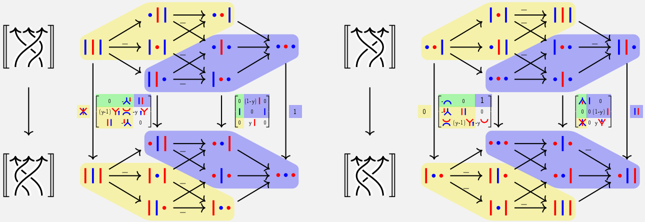
KhR_N is invariant under the sweeparound move, thus functorial in S^3.

Proving the sweeparound move

1. Reduce to the case of almost braid closures
2. Compare front and back versions of



3. Consider filtration by homological degree of extra crossings
4. Front and back versions of R1, R2, R3 agree* in associated graded

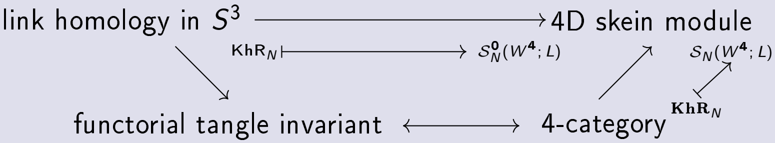


Break

Summary

- Strategy: generalise link homologies to skein modules of 4-manifolds
- Review: Khovanov–Rozansky $\mathfrak{gl}_N$ link homology KhR_N
- Theorem: KhR_N is functorial in S^3

What's next



From link homology to skein module

In analogy to

$$\mathrm{Sk}_N(M^3; P) := \frac{\mathbb{Z}[q^{\pm 1}]\langle \text{framed, oriented tangles in } (M^3, P)\rangle}{\langle \ker RT_N \text{ in } B^3 \hookrightarrow M^3\rangle}$$

we would like to define $\mathcal{S}_N^0(W^4; L)$ as:

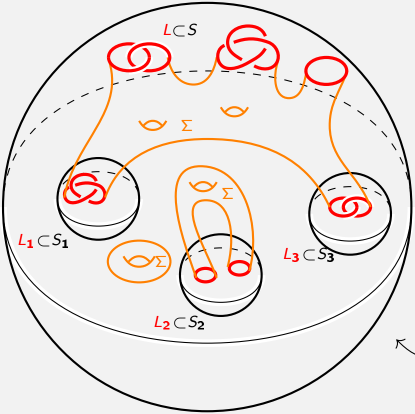
$$\frac{\Bbbk\langle \text{framed, oriented surfaces in } (W^4, L)\rangle}{\langle \ker [\![-]\!]_N \text{ in } B^4 \hookrightarrow W^4\rangle}$$

Problem: Want $\mathcal{S}_N^0(B^4; L) \cong \mathrm{KhR}_N(L)$, but this is not always spanned by images of cobordisms maps.

$\implies$ consider decorated framed, oriented surfaces.

Lasagna algebra

Khovanov–Rozansky homology is an algebra for the lasagna operad



$$\to \begin{pmatrix} \mathrm{KhR}_N(S^3, L) \\ \uparrow \mathrm{KhR}(D) \\ \bigotimes_i \mathrm{KhR}_N(S_i, L_i) \end{pmatrix}$$

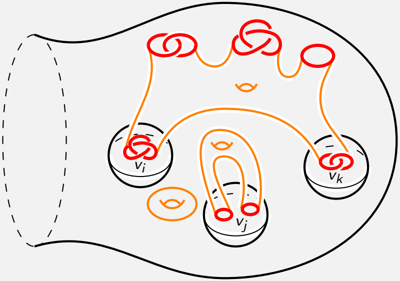
A lasagna diagram D with

L_i, L : input/output links

Σ : f., o. surface in $(B^4 \setminus \sqcup_i B^4_i; L \sqcup_i L_i)$

Khovanov–Rozansky skeins

A lasagna filling of W^4 with a link $L \subset \partial W^4$ is the data of:



B_i^4 : finitely many disjoint 4-balls in $W^{4\circ}$
L_i : input links in ∂B_i^4
Σ : f., o. surface in $(W^4 \setminus \sqcup_i B_i^4; L \sqcup_i L_i)$
$v_i \in \mathrm{KhR}_N(\partial B_i^4, L_i)$

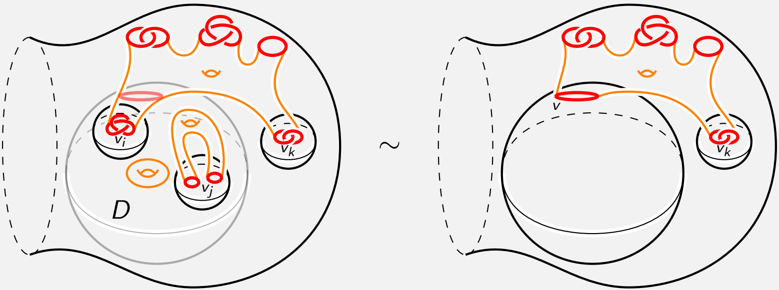
Definition of $\mathcal{S}_N^0(W^4; L)$

Definition

We define the bigraded vector space

$$\mathcal{S}_N^0(W^4; L) := \Bbbk\langle \text{lasagna fillings of } (W^4, L)\rangle / \sim$$

where the equivalence relation $\sim$ is generated by



with $v = \mathrm{KhR}(D)(v_i \otimes \cdots \otimes v_j)$.

Ribbon 2-category via KhR_N for tangles

$$\left\{\begin{matrix}\text{tangle diagrams}\\ \text{movies of diagrams/m. moves}\end{matrix}\right\} \xrightarrow{[\![-]\!]_N} H^*\mathrm{Ch}^b(N\mathrm{Foam})$$

$$\updownarrow \cong \qquad\qquad \downarrow \chi_q$$

$$\left\{\begin{matrix}\text{tangles embedded in } B^3\\ \text{cobordisms in } B^3 \times I/\text{isotopy}\end{matrix}\right\} \xrightarrow{RT_N \circ K_0} \mathrm{Rep}(U_q(\mathfrak{gl}_N))$$

Theorem (M.-W.-W. 2019)

There is a braided monoidal dg 2-category $\mathbf{KhR}_N$ with

- Objects: tangle boundary sequences
- 1-morphisms: Morse data for tangle diagrams
- 2-morphisms: $\mathbf{KhR}_N(T_1, T_2) := H^*\mathrm{Ch}^b(N\mathrm{Foam})([\![T_1]\!]_N, [\![T_2]\!]_N)$.

Think of $\mathbf{KhR}_N$ as categorification of $\mathrm{Rep}(U_q(\mathfrak{gl}_N))$.

Towards derived skein modules & TFT

Questions

Is $\mathbf{KhR}_N$ 4-dualizable and $SO(4)$-fixed in a suitable 5-category of braided monoidal dg 2-categories? What is the role of the sweeparound move?

$\implies$ a local 01234ε-D oriented TFT via the cobordism hypothesis.

Proposed direct construction for the 4ε part:

Theorem (M.-W.-W. 2019)

KhR_N controls a disk-like 4-category, determines $\mathcal{S}_N(W^4; L)$ via the blob complex (Morrison–Walker).

Examples

Example (B^4)

$\mathcal{S}_N(B^4; L) \cong \mathcal{S}_N^0(B^4; L) \cong \mathrm{KhR}(L)$ from the definition.

Example ($B^3 \times S^1$)

$\mathcal{S}_N(B^3 \times S^1; L)$ is the Hochschild homology of a dg category $\mathcal{S}_N(B^3; pts)$ with coefficients in a dg bimodule associated to a tangle closing to L.

Conjecture

$\mathcal{S}_N(B^3 \times S^1; L)$ is a $\mathfrak{gl}_N$-analog of Rozansky's homology theory for null-homologous links L in $S^2 \times S^1$. Computable as $\mathcal{S}_N(B^4; L')$ for L' obtained by infinite full-twist insertion. C.f. Willis.