Bott Periodicity via Simplicial Spaces

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TO NATHAN JACOBSON ON HIS 70TH BIRTHDAY

1. INTRODUCTION

We give a proof of complex Bott periodicity that uses some of the general ideas of Segal and Quillen on classifying spaces [1, 2, 4]. We attempt to understand which essential properties of the complex numbers and the unitary group are responsible for the periodicity, as contrasted with general facts on classifying spaces and linear groups over general topological rings (i.e., we compare complex K-theory and general algebraic K-theory). From this point of view the only essential properties of the complex numbers are: the "spectral theorem" ($n \times n$ unitary matrices can be diagonalized) and the fact that the Stiefel manifold of k-frames in $\mathbb{C}^N$ is highly connected for large N.

2. The periodicity theorem to be proved is that $AU = \mathbb{Z} \times BU$ or equivalently $B(\mathbb{Z} \times BU) = U$ for a suitable monoid structure on $\mathbb{Z} \times BU$. Since this last monoid structure is not easy to describe directly we replace $\mathbb{Z} \times BU$, the space representing $K(X)$, by the space representing the monoid $\mathrm{Vect}(X)$ of vector bundles and Whitney sum, namely, the disjoint union $\coprod_{n>0} BU_n$ ($BU_0 = $ point). The maps $BU_m \times BU_n \to BU_{m+n}$ induced by $U_m \times U_n \hookrightarrow U_{m+n}$ give $\coprod_n BU_n$ its monoid structure. The Group Completion Theorem [1; 4, Section 7, 6] implies that the loops on the classifying space of the monoid $\coprod_n BU_n$ are $\mathbb{Z} \times BU$, i.e., $B(\coprod_n BU_n) = B(\mathbb{Z} \times BU)$: this is a general theorem valid for linear groups over arbitrary topological rings. Periodicity is then the statement

$$B\left(\coprod_n BU_n\right) = U.$$

We shall first prove that $\coprod_n BU_n$ can be replaced by the disjoint union of Grassmann manifolds of n-planes in $\aleph_0$ dimensional space: $Gr = \coprod_n Gr_n$; further in Gr we will have a partially defined multiplication and consequently

a classifying space $B \, Gr$ (made up out of flag manifolds) which we will show to be homotopy equivalent to $B(\coprod_n BU_n)$; finally $B \, Gr$ will be shown to be homeomorphic to U. Let $\mathbb{C}^\infty$ denote a vector space of countable infinite dimension over $\mathbb{C}$ and with hermitian inner product. U will denote the unitary transformations of $\mathbb{C}^\infty$ which are the identity on a subspace of finite codimension. Gr is the set of hermitian projections E of finite rank $\geqslant 0$: $E = E^* = E^2$. $B \, Gr$ is defined as the geometric realization $|X_\cdot|$ of the following simplicial space $X_\cdot = \{X_n \, ; \, n \geqslant 0\}$ $X_0 = $ point, $X_1 = Gr$

$$X_n = \{[E_1, ..., E_n] \in (Gr)^n \mid E_i E_j = 0 \text{ for } i \neq j\}.$$

Thus X_n is a flag manifold.

Face maps $X_n \to$ are given by

$$[E_1, ..., E_n] \mapsto [E_1, ..., E_i + E_{i+1}, E_{i+2}, ..., E_n] \quad (1 \leqslant i < n)$$

or by omitting E_1 or E_n (same as in the bar construction). Degeneracies consist of inserting the zero projection. We will show $|X_\cdot| \cong B(\coprod_n BU_n)$; granting this for the moment the main step is:

THEOREM $|X_\cdot|$ *is homeomorphic to* U.

Proof. $|X_\cdot| = \coprod_n X_n \times \Delta^n$ modulo an equivalence relation. Δ^n is the standard n-simplex with points $\mathbf{t} = (t_1, ..., t_n)$ where $1 \geqslant t_1 \geqslant t_2 \geqslant \cdots \geqslant t_n \geqslant 0$. Map $X_n \times \Delta^n$ to U by

$$([E_1, ..., E_n], (t_1, ..., t_n)) \to e^{2\pi i (t_1 E_1 + \cdots + t_n E_n)}.$$

The equivalence relation is preserved and the map is a homeomorphism by the spectral theorem.

3. Let $Y_\cdot$ be the simplicial space associated to the monoid $\coprod_n BU_n$, i.e., $Y_0 = $ point, $Y_q = (\coprod_n BU_n)^q$ so that $|Y_\cdot| = B(\coprod_n BU_n)$. We want to show $|X_\cdot|$ is homotopy equivalent to $|Y_\cdot|$; along the way we will prove BU_n is homotopy equivalent to $Gr(n) = \{E \mid \text{tr} \, E = n\}$. To do this we construct an intermediate simplicial space $Z_\cdot$ and maps

$$X_\cdot \xleftarrow{\;r\;} Z_\cdot \xrightarrow{\;d\;} Y_\cdot$$

such that $|Z_\cdot| \xrightarrow{|r|} |X_\cdot|$ and $|d_\cdot| : |Z_\cdot| \to |Y_\cdot|$ are homotopy equivalences. The following general criterion will be used (see appendixes in [1, 2, 5]: a map of simplicial spaces $Z_\cdot \to X_\cdot$ (satisfying a very mild condition on each space) such that for each n, $Z_n \to X_n$ is a homotopy equivalence, induces a homotopy equivalence $|Z_\cdot| \to |X_\cdot|$. To construct $Y_\cdot, Z_\cdot, X_\cdot$ we begin with certain categories C, F, G each of which will have a "composition": if $\mathfrak{C}$ denotes the

category then we will have in each case a subcategory $\mathfrak{C}_n$ of the product category $\mathfrak{C}^n$ and functors $\mathfrak{C}_n \to \mathfrak{C}_m$ defining in fact a simplicial category $\mathfrak{C}_.$. Then the sequence of classifying spaces $\{B\mathfrak{C}_n \; ; \; n = 0, 1,...\}$ will be the simplicial space $Y.$ or $X.$ or $Z.$. The first category, C, has objects the integers $n \geqslant 0$; the morphisms go from n to n and consist of the unitary automorphisms of $\mathbb{C}^n$. C_n is just the cartesian product C^n ($=$ point for $n = 0$) and $Y. = \{Y_n\}$ is defined as $Y_0 =$ point, $Y_1 = BC = BC_1 = \coprod_k BU_k$, $Y_n = (\coprod_k BU_k)^n$ so that $| Y. | = B(\coprod_k BU_k)$. The multiplication on $\coprod_k BU_k$ defines $Y_n \to Y_m$ for $m < n$ in the usual way.

The category F has as objects n-frames in $\mathbb{C}^\infty$, i.e., injective isometric maps $\theta \colon \mathbb{C}^n \to \mathbb{C}^\infty$ ($n = 0, 2,...$) a morphism $\alpha \colon \theta \to \theta'$ is defined if and only if θ, θ' are both n-frames with the *same image* in $\mathbb{C}^\infty$ and consists of the *unique* isometry α of $\mathbb{C}^n$ such that $\theta' \circ \alpha = \theta$. The category F_n is the subcategory of F^n with objects the n-tuples $(\theta_1 ,..., \theta_n)$ of objects of F such that the images of θ_i and θ_j are mutually orthogonal for $i \neq j$; functors $F_n \to F_{n-1}$ are obtained by *adding* θ_i and θ_{i+1} ($i = 1,..., n - 1$) or omitting θ_1 or θ_n (as in the bar construction). Degeneracies $F_n \to F_{n+1}$ are given by inserting the frame $\theta = 0$. A functor $d \colon F \to C$ is given by $d(\theta) = n$ if θ has domain $\mathbb{C}^n$, and $d(\alpha) = \alpha$ (α is a morphism) and similarly for F_n, C_n . Thus F, C differ only in their objects: for each n, C has one object whereas F has as objects the Stiefel manifold $V_{n,\alpha}$ of all n-frames in $\mathbb{C}^\infty$, which is a *contractible space*. The classifying space BF is constructed from the simplicial space whose q-simplices are the sequences $(\theta, \alpha_1 ,..., \alpha_q)$ where $\theta \colon \mathbb{C}^n \to \mathbb{C}^\infty$ and $\alpha_i \in U_n$; the corresponding simplex for BC (under the map d) is $(\alpha_1 ,..., \alpha_q)$: thus the spaces of q-simplices for BF, BC are: disjoint union over all n of $V_{n,\infty} \times (U_n)^q$, respectively $(U_n)^q$, and the map given by d is projection on the second factor, thus a homotopy equivalence for each q. It follows that $BF = Z_1 \to BC = X_1$ is a homotopy equivalence. In exactly the same way, $BF_q = Z_q \to BC_q = X_q$ is a homotopy equivalence so that finally $| Z. | \to | X. | = B(\coprod_n BU_n)$ is a homotopy equivalence.

The category G with $BG = Gr$ has objects the elements of Gr and morphisms only the identity morphisms (one for each object), so that $BG =$ space of objects $= Gr$. The category $G_0 =$ point, $G_1 = G$; G_n has objects the n-tuples $(E_1 ,..., E_n)$ of objects of G which are mutually orthogonal and morphisms only the identities. $BG_n = X_n$; the face and degeneracy functors $G_m \to G_n$ were previously described.

The functor $r \colon F \to G$ takes the object θ to the projection on the image of θ; and similarly for functors $F_n \to G_n$. For each point E of $BG_1 = BG$ the inverse image of E under the map $B(r) \colon BF \to BG$ consists of the classifying space of the subcategory of F of all frames θ with $r(\theta) = E$ as objects, and all "changes of frames" α as morphisms; this category of "frames for E" has a unique morphism between each two objects and so its classifying space is contractible (it may be identified with the total space E_{U_n} of the universal principal U_n bundle, where $n = \operatorname{tr} E$). For each n, if $BF(n)$ and $BG(n)$ denote the components corresponding to projections of rank n, then $B(r) \colon BF(n) \to BG(n)$ is a locally

trivial fibration with contractible fiber; thus $BF \to BG$ is a homotopy equivalence. In exactly the same way $BF_q \to BG_q$ is a homotopy equivalence (F_q, G_q are categories of q-tuples), and finally $| r | : | Z . | \to | X . |$ is a homotopy equivalence. This concludes the proof that $B(\coprod_n BU_n)$ is homotopy equivalent to U.

4. We indicate briefly how the Group Completion Theorem can be avoided, or rather replaced by the simpler Proposition 1.4 of [1] which states, in the context of interest to us, that if $\mathfrak{C}$. is a simplicial category of the type considered above such that $B\mathfrak{C}_2 \xrightarrow{p_1 \times p_2} B\mathfrak{C}_1 \times B\mathfrak{C}_1$ is a homotopy equivalence (p_1, $p_2 : B\mathfrak{C}_2 \to B\mathfrak{C}_1$ being the "projections" induced by the face maps $[0, 1] \to [0, 1, 2]$ taking 0, 1 to 0, 1 respectively, to 1, 2) and if $m: B\mathfrak{C}_2 \to B\mathfrak{C}_1$ denotes the "composition" (induced by the face map $[0, 1] \to [0, 1, 2]$ taking 0 to 0, 1 to 2) and if the map

$$B\mathfrak{C}_1 \times B\mathfrak{C}_1 \xrightarrow{(p_1 \times p_2)^{-1}} B\mathfrak{C}_2 \xrightarrow{m} B\mathfrak{C}_1$$

makes $B\mathfrak{C}_1$ an H-space with homotopy inverse, then $B\mathfrak{C}_1$ is the loop space on the geometric realization $| B\mathfrak{C} . |$. In this way of proceeding we also work directly with $\mathbb{Z} \times BU$ (rather than $\coprod_n BU_n$) and construct a simplicial space whose realization is $B(\mathbb{Z} \times BU)$ and is homeomorphic to $U \times U/U$ (that is, the quotient of $U \times U$ by the relation $(u_1, u_2) \sim (u_1 u, u_2 u)$ where $u_1 - I$, $u_2 - I$, and $u - I$ have mutually orthogonal supports). To construct $\mathbb{Z} \times BU$ we start with the category PG whose objects are arbitrary pairs (E, E') of projections on $\mathbb{C}^\alpha$ and with unique morphism from (E_1, E_1') to (E_2, E_2') given by a projection F orthogonal to both E_1, E_1' such that $E_1 + F = E_2$, $E_1' + F = E_2'$. We could similarly define categories of pairs PF, PC to show that we obtain the same classifying space for PG and PC where PC has objects pairs (m, m') and morphisms (m, m') to (n, n') pairs of injective isometries $\alpha: \mathbb{C}^m \to \mathbb{C}^n$, $\alpha': \mathbb{C}^{m'} \to \mathbb{C}^{n'}$ together with isometry β from the orthogonal complement of Im α to that of Im α' (a category used by Quillen for general rings). Instead we will just remark that the classifying space of PG at least intuitively classifies pairs of vector bundles modulo addition of a common bundle: i.e., we claim $BPG = \mathbb{Z} \times BU$.

Further categories PG_n have objects n-tuples of pairs $[(E_1, E_1'),..., (E_n, E_n')]$ where $E_1,..., E_n$ are orthogonal, and $E_1',..., E_n'$ are orthogonal, and morphisms n-tuples $(F_1,..., F_n)$ (F_i orthogonal to E_i, E_i' and $E_i + F_i$ orthogonal for $i = 1,...$ n and similarly for $E_i' + F_i$).

We can then describe a homeomorphism $| BPG . |$ to $U \times U/U$, only the notation being cumbersome. The functor $G \to PG$, $E \mapsto (E, 0)$ on objects, then corresponds to $U \to U \times U/U$, $u \mapsto (u, 1)$.

5. Finally we note that if we replace the complex numbers throughout by reals, then we have a proof that $B(\coprod_n BO_n) = U/O$. However the complete real periodicity requires something more.

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