# HOPF RINGS IN ALGEBRAIC TOPOLOGY

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ABSTRACT. These are colloquium style lecture notes about Hopf rings in algebraic topology. They were designed for use by non-topologists and graduate students but have been found helpful for those who want to start learning about Hopf rings. They are not "up to date," nor are then intended to be, but instead they are intended to be introductory in nature. Although these are "old" notes, Hopf rings are thriving and these notes give a relatively painless introduction which should prepare the reader to approach the current literature.

This is a brief survey about Hopf rings: what they are, how they arise, examples, and how to compute them. There are very few proofs. The bulk of the technical details can be found in either [RW77] or [Wil82], but a "soft" introduction to the material is difficult to find.

Historically, Hopf algebras go back to the early days of our subject matter, homotopy theory and algebraic topology. They arise naturally from the homology of spaces with multiplications on them, i.e. $H$-spaces, or "Hopf" spaces. In our language, this homology is a group object in the category of coalgebras. Hopf algebras have become objects of study in their own right, e.g. [MM65] and [Swe69]. They were also able to give great insight into complicated structures such as with Milnor's work on the Steenrod algebra [Mil58]. However, when spaces have more structure than just a multiplication, their homology produces even richer algebraic stuctures. In particular, with the development of generalized cohomology theories, we have seen that the spaces which classify them have a structure mimicking that of a graded ring. The homology of all these spaces reflects that fact with a rich algebraic structure: a Hopf ring, or, a ring in the category of coalgebras. This then is the natural tool to use when studying generalized homology theories with the aim of developing them to the point of being useful to the average working algebraic topologist. Hopf rings lead to elegant descriptions of the answers and new techniques for computing them.

The first known reference to Hopf rings was in Milgram's paper [Mil70]. Lately, the theoretical background for Hopf rings has been greatly expanded, [Goe99] and [HT98], i.e. Hopf rings have begun to be studied as objects in their own right, and more and more applications are being found for them [BJW95] [BW] [Goe99] [HH95] [Kas] [Har91] [Tur97] [ETW97] [KST96] [Tur93] [Kas94] [Kas95] [HR95] [GRT95] [RW96].

Although these notes have been around for awhile, the demand for them has been increasing and so it seemed it might be appropriate to publish them as a survey. They are intended to be readable to all in the way that colloquium lectures should be understandable to graduate students and those in other fields entirely. However, their main use seems to be to serve as an introduction to the material for algebraic topologists so that both old and new technical papers can be approached with some confidence. As an introduction, it is not intended, nor is it necessary, that these notes be "up to date." This very lack of completeness adds a great deal to their readability.

A *topologist* studies topological spaces and continuous maps. The typical example of a nice topological space is a *CW complex* ; i.e., a space built up from cells. Let us say we have built $X$. We can add an $n$-dimensional cell, $D^n$, to $X$ using any continuous map

$$f : S^{n-1} \cong \partial D^n \to X;$$

just glue $D^n$ to $X$ by identifying $x \in S^{n-1}$ with $f(x) \in X$. We get a space $Y = XU_f D^n$. In this way we can build a large class of topological spaces which have a certain amount of geometric intuition behind them.

A *homotopy theorist* feels that there are far too many topological spaces and continuous maps to deal with effectively. Perhaps this is a cowardly approach, but we immediately put an equivalence relation on the continuous maps from $X$ to $Y$. We say $f \sim g$, *$f$ is homotopic to $g$*, if $f$ can be continuously deformed into $g$ ; i.e., if there is a continuous map $F : X \times I \to Y$, $(I = [0, 1])$, such that $F \mid X \times 0 = f$ and $F \mid X \times 1 = g$. This is an equivalence relation and we denote the set of equivalence classes by $[X, Y]$ and call this the *homotopy classes of maps from $X$ to $Y$*. We say $X$ and $Y$ are *homotopy equivalent* or of the *same homotopy type* if we have maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim I_X$ and $f \circ g \sim I_y$. Two fundamental problems in homotopy theory are to determine if $X$ and $Y$ are of the same homotopy type and to compute $[X, Y]$.

Other than being too cowardly to approach all topological spaces and maps at once, what other good reasons are there for studying homotopy theory? The main reasons are that homotopy theory has a strong give and take with many other areas of mathematics, particularly geometry and algebra. plus some intrinsically interesting questions of its own. Most of these lectures will be concerned with the algebraic aspects of homotopy theory but first we give some geometric applications.

We look first at vector bundles over a space $X$. A *vector bundle* assigns a vector space to every point of $X$. This is done in a continuous fashion. The $k$-dimensional vector bundles over $X$ are equivalent to the homotopy classes of maps from $X$ to a fixed space $BO_k$ ; $[X, BO_k]$, [Ste51]. So, as is the case with many geometric problems, the classification of isomorphism classes of $k$-dimensional vector bundles is reduced to the computation of homotopy classes of maps. Furthermore, it is clear that studying $BO_k$ is very useful for this problem. It comes about by a standard construction which builds a classifying space, $BG$, for any group $G$. This is just the special case of the $k$-th orthogonal group of $k \times k$ matrices.

We can push this example even further to give one of the deepest applications of homotopy theory to mathematics. We consider $n$-dimensional smooth compact manifolds, $M^n$. In the 1940's Whitney showed that any such manifold immersed in $R^{2n}$, [Whi44]. An immersion is just a differential map to the Euclidean space such that *df* is injective on the tangent bundle, $\tau$ , at every point. Any immersion $M^n \hookrightarrow \mathbb{R}^{n+k}$ gives rise to a $k$-dimensional *normal bundle*, $\nu$. It has the property that if you add $\nu$ to $\tau$ as bundles, the sum is $M^n \times R^{n+k}$. The normal bundle must come from a map $M^n \to BO_k$, and be the pull-back of the *universal bundle*, $\xi_k$ , over $BO_k$. In particular, Whitney's theorem says you always can have an $n$-dimensional normal bundle for $M^n$, $M^n \to BO_n$. Hirsch and Smale, [Hir59]and [Sma59], reduced the geometric question for immersions to homotopy theory by showing that if this map to $BO_n$ lifted to $BO_k$, $k < n$ , then $M^n \hookrightarrow \mathbb{R}^{n+k}$.

$$\begin{array}{ccccc} & & BO_k & \longleftarrow & \xi_k \oplus \mathbb{R}^{n-k} \\ & \nearrow & \downarrow & & \downarrow \\ M^n & \longrightarrow & BO_n & \longleftarrow & \xi_n \end{array}$$

The map from $BO_k$ to $BO_n$ comes from classifying the bundle $\xi_k \oplus R^{n-k}$. One of the great theorems of homotopy theory is the result of Ralph Cohen, [Coh85], that all $n$-dim manifolds, $M^n$, immerse in $\mathbb{R}^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of ones in the binary expansion of $n$; i.e., $\alpha(n) = \Sigma a_i$ where $a_i = 0$ or 1, and $n = \Sigma a_i 2^i$. This conjecture has been around for a long time. Brown and Peterson, [BP79], constructed a space $X_n$ and a map $X_n \to BO$ $(BO_\infty)$ such that the conjecture is true if this map lifts

$$\begin{array}{ccc} & & BO_{n-\alpha(n)} \\ & \nearrow & \downarrow \\ X_n & \longrightarrow & BO \end{array} .$$

R. Cohen finished the theorem from there.

Since manifolds are a great meeting ground for all mathematics, this is a theorem with content that any mathematician can appreciate.

Because so many geometric properties can be classified by homotopy theory there is sometimes a feeling that homotopy theory is just a service department for other areas of mathematics. It does, however, interact on other levels with other fields.

Our next reason for going to homotopy theory is that it is much more accessible to algebraic techniques. Therefore we move on now to *algebraic topology.* Here we want to have a rule that assigns some algebraic object to every space; $X \longmapsto E_*X$. This may just be a set, or have some complicated algebraic structure: groups, rings, algebras, etc. For every map $f : X \to Y$ we want a corresponding algebraic map $f_* : E_*X \to E_*Y$. If $f \sim g$ we want $f_* = g_*$. However, this property usually holds for most algebraic invariants anyway, so we are usually forced to go to homotopy theory if we use these algebraic techniques. (We also use algebraic objects $E^*X$ where the algebraic map reverses direction from that of the topological map.) So if $f_* \neq g_*$ then $f \not\sim g$. If $X \sim Y$, then $E_*X \cong E_*Y$. The more algebraic theories we have, the better the chance of distinguishing two maps. The richer the algebraic structure, the more difficult it is to have an isomorphism. For example, it is much easier for two sets to be the same than for them to be isomorphic as groups or rings. Of course a third thing we want is computability, which we usually do not have. It is much easier to define algebraic invariants than to compute them.

Our first examples of algebraic invariants of this sort are the usual mod 2 homology, $H_*X$, and cohomology, $H^*X$. They both satisfy our homotopy condition. The cohomology is the dual to the homology which is just a collection of $Z/2$ vector spaces $H_*X = \{H_iX\}_{i\geq 0}$. $H_nX$ is defined using maps of generalized $n$-dimensional triangles into $X$. In particular, $H_nX$ tells us something about how the $n$-cells of $X$ are related to the $n+1$ dimensional cells and the $n-1$ dimensional cells. Of course the first thing we do is invent homological algebra to deal with homology and get away from the geometry.

The cohomology has more structure than just a collection of vector spaces. Applying $H^*(-)$ to the diagonal map $\triangle : X \to X \times X$ we get

$$H^*X \leftarrow H^*(X \times X) \cong H^*X \otimes H^*X \ .$$

The isomorphism is the Künneth theorem but we must define the tensor product for this to make sense. We have

$$H^n(X \times X) \cong \bigoplus_{i+j=n} H^iX \otimes H^jX \ .$$

What we have now is an algebra, or more precisely, a *graded algebra*. It allows us to multiply $x_i \in H^iX$ and $x_j \in H^jX$ to get an element $x_ix_j \in H^{i+j}X$. This gives us a richer, stronger structure of the sort we want. It is easy to find $X$ and $Y$ with $H^*X \cong H^*Y$ as collections of vector spaces but not as graded algebras. Dually, the homology also has a more complex structure. The diagonal gives

$$\begin{array}{ccc} H_*X & \longrightarrow & H_*(X \times X) \cong H_*X \otimes H_*X \\ x & \xrightarrow{\psi} & \Sigma x' \otimes x'' \end{array}$$

with the degree of $x =$ sum of degree of $x'$ and $x''$, we call this structure a *coalgebra*. It contains the same information as the dual algebra in cohomology contains.

For our next algebraic example we use real $K$-theory $KO(X)$, [Ati89]. To define this we let two vector bundles $\xi$ and $\eta$ be equivalent if $\xi \oplus R^k \cong \eta \oplus R^k$. We can add vector bundles just like vector spaces. Now we formally insert additive inverses for vector bundles. Then there is a space, $BO$, such that

$$KO(X) \cong [X, Z \times BO],$$

where $Z$ is the set of integers with discrete topology. $KO(X)$ is an abelian group because we can now add and subtract our bundles. (It is abelian because $V \oplus W \cong W \oplus V$ for vector spaces.) We can also multiply two vector spaces by taking the tensor product. This gives us a product in $KO(X)$ and turns $KO(X)$ into a ring. A major theme in these lectures is that all structure must be reflected in structure on the classifying space. In this case, $Z \times BO$ must be a ring in some suitable sense. First there must be a product

$$\oplus : (Z \times BO) \times (Z \times BO) \longrightarrow Z \times BO.$$

We can construct this geometrically along with the construction of BO or we can use a general nonsense proof using Brown's representability theorem, [Bro62]. Brown's theorem states that if you assign some algebraic object $F(X)$ to every space $X$ and it has certain homotopy properties, then there is a space $F$ such that

$$F(X) \cong [X, F],$$

and furthermore, all algebraic properties of $F(X)$ are reflected in $F$. In our case, if we have two elements of $KO(X)$, $f$ and $g$, and we want to add then we can do it simply by the following diagram

$$\begin{array}{c} \oplus : (Z \times BO) \times (Z \times BO) \longrightarrow Z \times BO \\ \uparrow{\scriptstyle f \times g} \\ X \times X \\ \uparrow{\scriptstyle \Delta} \\ X \end{array}$$

The same discussion goes through for the ring map obtained from $\otimes$;

$$\otimes : (Z \times BO) \times (Z \times BO) \to Z \times BO.$$

$Z \times BO$ does not really become a ring, or even a group. If we write down diagrams that we would want to commute, for the inverse, commutativity or associativity, then these diagrams only commute up to homotopy! Likewise for the ring structure. We say $Z \times BO$ is a ring in the homotopy category. The basis for distributivity is the vector space isomorphism

$$V \otimes (U \oplus W) \cong (V \otimes U) \oplus (V \otimes W).$$

In general, if $F(X) \cong [X, F]$ is a ring, then $F$ is a ring, up to homotopy. We can now give our first example of a Hopf ring. For concreteness, think of $F$ as $Z \times BO$. We take the mod 2 homology $H_*F$. First, this is a coalgebra as it is with all spaces. Second, since $F$ is an abelian group (up to homotopy) we have a product

$$* : F \times F \to F$$

(corresponds to $\oplus$ for $F = Z \times BO$). Applying $H_*(-)$, we have $H_*F$ is a graded ring

$$* : H_*F \otimes H_*F \cong H_*(F \times F) \to H_*F.$$

This map is a map of coalgebras. This fact together with simultaneous algebra and coalgebra structures makes up a *Hopf algebra*, [MM65]. For us it would be better to call it a *coalgebraic group* because it is the structure which comes from the group structure of $F$, which in turn comes from the group structure of $F(X)$. We should think of this $*$ product as our "addition".

From the ring structure, $\circ : F \times F \to F$, of $F$ we get another map

$$\circ : H_*F \otimes H_*F \cong H_*(F \times F) \to H_*F$$

which we think of as "multiplication". Altogether we call this total structure on $H_*F$ a *Hopf ring*, or more appropriately, a *coalgebraic ring*. Such an object is a ring object in the category of coalgebras, much the same as $F$ is a ring in the homotopy category. Before we go into more detail we should give the distributivity law for Hopf rings. In this case it is (recall the coproduct of $x$ is $\Sigma x' \otimes x''$)

$$x \circ (y * z) = \Sigma (x' \circ y) * (x'' \circ z).$$

Hopf rings are our objects of study. They occur frequently in algebraic topology. In particular, anytime we define an algebraic invariant which is a ring and a homotopy invariant, then we can apply mod 2 homology to the classifying space and we have a Hopf ring.

We have seen the coalgebra structure show up in the distributivity law, but we have not really made clear the theoretical necessity of the coalgebra in the Hopf ring. It goes back to the definition of a group (or ring) object in an arbitrary category. When we define an ordinary group we must define an operation, given by a map

$$G \times G \to G\ .$$

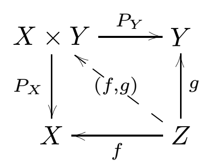
Well, here is the problem. What is that product on the left in an arbitrary category? The answer is that it does not always exist. For topological spaces $F(X)$ a group gives rise to a map on the classifying space level

$$F \times F \to F.$$

When we apply homology to this map we get

$$H_*F \otimes H_*F \cong H_*(F \times F) \to H_*F. \tag{1}$$

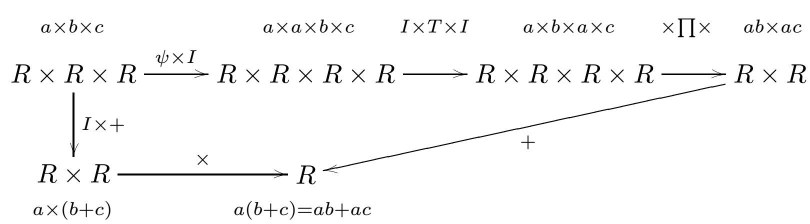
For this to be a group object in some sense then $H_*F \otimes H_*F$ must be the product of $H_*F$ with itself. However, this is not what we would choose as a "product" for graded vector spaces. Categorically speaking, what is a product? If we have $X$ and $Y$ in a category $C$, then the product $X \times Y$ is an object of $C$ with maps $P_X : X \times Y \to X$ and $P_Y : X \times Y \to Y$ such that for any other object $Z$ and maps $f : Z \to X$, $g : Z \to Y$, there is a unique map $(f, g)$ such that the diagram commutes:




For two graded vector spaces, $C_*$ and $D_*$, such as $H_*F$, the product is $(C_* \times D_*)_n = C_n \times D_n$. This has nothing to do with our information in (1). If our category is coalgebras, not graded vector spaces, then we have a different product entirely. If $C_*$ and $D_*$ are coalgebras, then $C_* \otimes D_*$ is the product! If $Z_*$ is a coalgebra with maps $f_* : Z_* \to C_*$ and $g_* : Z_* \to D_*$, we can define the map $(f, g)$ by

$$Z_* \xrightarrow{\psi} Z_* \otimes Z_* \xrightarrow{f \otimes g} C_* \otimes D_*\ .$$

This makes the coalgebras necessary. Now (1) works perfectly as it is a map of $H_*F \times H_*F \to H_*F$ in the category of coalgebras. It is now easy to see how the coproduct works into the distributivity law. For an ordinary ring, $R$, distributivity, $a(b + c) = ab + ac$, can be written in the form




The use of the diagonal map, $a \to a \times a$, translates to our new category as the coalgebra coproduct, and our new style distributivity law follows.

Having gone to so much trouble to say that $H_*(Z \times BO)$ (mod 2 homology) is a Hopf ring, we should describe it in detail. $Z \times BO$ is a familiar space to all topologists and the homology is well known to most, so this is an example of a Hopf ring in a familiar setting. First, $H_*(Z \times BO)$ is isomorphic to $H_*(Z) \otimes H_*BO$. $H_*Z$ is just $Z/2$ [Z], the group ring of $Z$ over $Z/2$. However, we now keep all of the structure of $Z$, so $H_*(Z)$ is the "ring-ring" of $Z$ over $Z/2$. It is a $Z/2$ vector space with basis given by all $[n]$, $n \in Z$. We compute the three operations as

$$\begin{aligned} \psi([n]) &= [n] \otimes [n] \ , \\ [n] * [m] &= [n+m] \ \text{ and} \\ [n] \circ [m] &= [nm] \ . \end{aligned}$$

The Hopf algebra structure of $H_*BO$ is not as familiar as that of $H^*BO$, but it is still very simple. We begin with $RP^\infty$ (the infinite real projective space). $H_iRP^\infty$, $i \geq 0$ is a $Z/2$ with generator $\beta_i$. We take the usual line bundle, $\eta$, over $RP^\infty$. Subtract the formal inverse of the trivial line bundle. This gives a map

$$\eta - 1 : RP^\infty \longrightarrow BO.$$

Let the image of $\beta_i$ be $b_i \in H_iBO$. Now it is known that $H_*BO$ is a polynomial algebra on the $b_i$, $i > 0$. The coproduct comes from $RP^\infty$ (dual to a polynomial algebra on one generator), i.e., $b_n \to \Sigma_{i+j=n} b_i \otimes b_j$. This completely describes the Hopf algebra, or "group", structure of $H_*(Z \times BO)$. The only new thing we are talking about is the $\circ$ product, i.e., the Hopf ring, or "ring", structure. Chasing down this structure will give us some practice with Hopf rings. We need some notation. Define $\beta(s) = \Sigma \beta_i s^i$ in $H_*RP^\infty[[s]]$. The coproduct of $H_*RP^\infty$ patches together to give $\beta(s) \to \beta(s) \otimes \beta(s)$. Now we know that $RP^\infty$ has a product on it making $H_*RP^\infty$ an algebra. In fact $H_*RP^\infty$ is a divided power algebra, i.e., $\beta_i\beta_j = (i,j)\beta_{i+j}$ where $(i,j)$ is the binomial coefficient. In our new notation this is just $\beta(s)\beta(t) = \beta(s+t)$. To see this, just look at the coefficient of $s^it^j$ on both left, $\beta_i\beta_j$, and right, $\beta_{i+j}(i,j)$. The product on $RP^\infty$ fits into a commuting diagram with $Z \times BO$

$$\begin{array}{ccc} RP^\infty \times RP^\infty & \longrightarrow & RP^\infty \\ \downarrow & & \downarrow \\ \otimes : (X \times BO) \times (Z \times BO) & \longrightarrow & Z \times BO \ . \end{array}$$

Unfortunately, the map $RP^\infty \to Z \times BO$ is not the one discussed earlier, $\eta - 1$, but it goes to the first component, $1 \times BO$, not $0 \times BO$. It is just the map $\eta$, so we can get it on homology by "adding" 1, that is $\beta(s) \to b(s) * [1]$. So, applying homology to the above diagram we get

$$\begin{array}{ccc} \beta(s) \otimes \beta(t) & \longrightarrow & \beta(s+t) \\ \downarrow & & \downarrow \\ (b(s) * [1]) \otimes (b(t) * [1]) & \longrightarrow & (b(s) * [1]) \circ (b(t) * [1]) = b(s+t) * [1] \ . \end{array}$$

To evaluate $b(s+t)$ we can apply $*[-1]$ to both sides. The right side is

$$b(s+t) * [1] * [-1] = b(s+t) * [0] = b(s+t).$$

The left side is more complicated. We begin with

$$((b(s) * [1]) \circ (b(t) * [1])) * [-1].$$

Let $x = b(s) * [1]$, $y = b(t)$ and $z = [1]$. We use the distributive law

$$x \circ (y * z) = \Sigma(x' \circ y) * (x'' \circ z).$$

We need to compute $\psi(x)$.

$$\begin{aligned}\psi(b(s)*[1]) &= \psi(b(s))*\psi([1]) = (b(s)\otimes b(s))*([1]\otimes[1]) \\ &= (b(s)*[1])\otimes(b(s)*[1])\ .\end{aligned}$$

So

$$(b(s)*[1])\circ(b(t)*[1]) = ((b(s)*[1])\circ b(t))*((b(s)*[1])\circ[1])\ .$$

The $\circ$ multiplication by $[1]$ is the "ring" unit. So $(b(s)*[1])\circ[1] = b(s)*[1]$. On the other part we use the distributivity law from the right.

$$\begin{aligned}(b(s)*[1])\circ b(t) &= (b(s)\circ b(t))*([1]\circ b(t)) \\ &= b(s)\circ b(t)*b(t)\ .\end{aligned}$$

Making our substitutions we have

$$\begin{aligned}((b(s)*[1])\circ(b(t)*[1]))*[-1] &= ((b(s)*[1])\circ b(t))*((b(s)*[1])\circ[1])*[-1] \\ &= (b(s)\circ b(t))*([1]\circ b(t))*(b(s)*[1])*[-1] \\ &= b(s)\circ b(t)*b(t)*b(s)\ .\end{aligned}$$

We have proven the relation

$$b(s)\circ b(t)*b(t)*b(s) = b(s+t).$$

Looking at the coefficients of $s^i t^j$, $i+j=n$, we see that $b_n$ can always be written in terms of lower $b_k$ and $\circ$ and $*$ unless $n$ is a power of 2. Thus $H_*(Z\times BO)$ can be described completely from the elements $b_{2^i}$. Most of the Hopf rings that we discuss will have this property: algebraically they are generated by very few elements. This is possible because we have two products we can use to construct more elements with.

The cohomology, $H^*(BO)$, is a polynomial algebra on the Stiefel-Whitney classes. If this is viewed as a module over the Steenrod algebra we know we only need to start with the $\omega_{2^i}$ also; but the Steenrod algebra is much more complicated than the structure I have been discussing.

Since these notes were first written a truly gruesomely detailed description of this Hopf ring has been obtained, [Str92].

Our next example of a Hopf ring comes again from our algebraic invariants for homotopy theory. We can also demonstrate various levels of richness in algebraic structure and show how this is reflected in the classifying spaces again.

Our example is just the mod 2 cohomology, $H^*X$. This is really a sequence of algebraic invariants and so requires a sequence of classifying spaces. We have that there exist spaces, $\mathbf{H}_n$, generally denoted $K(Z/2,n)$ and called Eilenberg–MacLane spaces, such that

$$H^n(X)\cong[X,\mathbf{H}_n]\ .$$

The $\mathbf{H}_n$ are fun spaces to study. Another property that they have which characterizes them is that

$$[S^k,\mathbf{H}_n] = \begin{array}{ll} 0 & \text{if} \quad k\neq n \\ Z/2 & \text{if} \quad k=n\ . \end{array}$$

Because $H^nX$ is a group (it is a vector space) we must have a map $\mathbf{H}_n\times\mathbf{H}_n\to\mathbf{H}_n$ which turns $\mathbf{H}_n$ into a group up to homotopy. As it happens, because Eilenberg–MacLane spaces are so basic, these spaces can actually be constructed as abelian groups, but they are the only such spaces which can be, [Mil67].

As we have discussed already, $H^*X$ is a graded ring. Its multiplication must be reflected in its classifying spaces, and so it is, with maps

$$\mathbf{H}_i \times \mathbf{H}_j \to \mathbf{H}_{i+j}$$

which can be used to define the product just as $\oplus$ and $\otimes$ were used with $K$-theory to define addition and multiplication. Before we look at the implication of this graded ring structure on our concept of Hopf rings we want to revert to our discussion of richer structure on our algebraic invariants.

Cohomology satisfies certain basic axioms which imply the fact about $[S^k, \mathbf{H}_n]$. These axioms also imply that $H^*X \cong H^{*+1}\Sigma X$, where $\Sigma X$ is the suspension of $X$. It is a homotopy theoretic fact that

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

where $\Omega Y$ is the loop space of $Y$ (i.e. the topological space of all maps of the unit interval into $Y$ which start and stop at the same "base" point). Combined, we get

$$[X, \mathbf{H}_n] \cong H^n X \cong H^{n+1}\Sigma X \cong [\Sigma X, \mathbf{H}_{n+1}] \cong [X, \Omega\mathbf{H}_{n+1}]$$

which can be used to show that $\Omega\mathbf{H}_{n+1} \cong \mathbf{H}_n$.

The cohomology $H^*X$ is a module over an algebra called the Steenrod algebra. This algebra is very complicated and the module structure gives us a much richer structure than only the cohomology algebra structure. Of course the Steenrod algebra, $A$, is a graded algebra and $H^*X$ is a *graded module*. The Steenrod algebra has a homotopy interpretation:

$$A^i \cong [\mathbf{H}_n, \mathbf{H}_{n+i}] \cong H^{n+i}\mathbf{H}_n, n > i.$$

Given a map $f : \mathbf{H}_{n+1} \to \mathbf{H}_{n+i+1}$ we can get another map $\Omega f : \Omega\mathbf{H}_{n+1} = \mathbf{H}_n \to \Omega\mathbf{H}_{n+i+1} \cong \mathbf{H}_{n+i}$. This is an isomorphism of maps for $n > i$. This allows us to think of the Steenrod algebra module structure as composition

$$X \to \mathbf{H}_n \to \mathbf{H}_{n+i}$$

if we wanted to. In particular, the Steenrod algebra is not commutative.

More precisely, the Steenrod algebra is generated by elements $Sq^i \in A^i$, $i \geq 0$. With the relations, however, only the $Sq^{2^i}$ are needed to generate. It is possible to mix up the Steenrod algebra module structure of $H^*X$ and the algebra structure of $H^*X$ to make an even richer structure. We have that

$$Sq^n(xy) = \sum_{i+j=n} Sq^i(x)Sq^j(y) \ .$$

This looks a little familiar, and, in fact, can be used to put a coalgebra structure on $A$. $A$ becomes a Hopf algebra and $H^*X$ an algebra over the Hopf algebra $A$.

The coalgebra structure comes from the map $\mathbf{H}_n \wedge \mathbf{H}_n \to \mathbf{H}_{2n}$. This Hopf algebra structure on $A$ has many practical applications. First and foremost, the dual, $A_*$ is also a Hopf algebra. The algebra structure is commutative and $A_*$ is a polynomial algebra

$$A_* \cong Z/2[\xi_1, \xi_2, \dots] \ , \ \ |\xi_n| = 2^n - 1 \ .$$

The entire Hopf algebra structure is given by this and a simple coproduct formula for the $\xi_n$ ($\xi_n \to \sum \xi_j^{2^{n-j}} \otimes \xi_{n-j}$).

We now go back to our new example of a Hopf ring. We have already said that whenever we have a ring in algebraic topology we can get a Hopf ring by applying the mod 2 homology to the classifying space. In the case of $H^*X$ we have a graded ring, not just a ring. This will also give rise to a Hopf ring, but this time the Hopf ring will be a *graded Hopf ring.* We make this translation in two steps, as usual, with the classifying spaces in the middle. The mod 2 cohomology $H^*X$ is a graded ring, i.e., a collection of abelian groups $\{H^nX\}_{n\geq 0}$ with a (distributive, associative, commutative, etc.) product that pairs these groups, $H^iX \otimes H^jX \to H^{i+j}X$. In turn, the classifying spaces, $\mathbf{H}_* = \{\mathbf{H}_n\}_{n\geq 0}$, form a graded ring object in the homotopy category. That is, each $\mathbf{H}_n$ is a group in the sense already discussed and there is a pairing $\mathbf{H}_i \times \mathbf{H}_j \to \mathbf{H}_{i+j}$ between these "groups" which make all the appropriate diagrams commute up to homotopy. Applying mod 2 homology we get $H_*\mathbf{H}_*$ is a graded ring object in the category of coalgebras, that is, a collection of "abelian groups" (bicommutative Hopf algebras), $\{H_*\mathbf{H}_n\}_{n\geq 0}$. This "addition" we denote by $*$. The "graded ring" structure is a collection of pairings

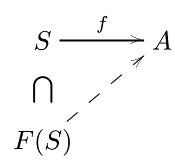
$$\circ : H_*\mathbf{H}_i \otimes H_*\mathbf{H}_j \to H_*\mathbf{H}_{i+j} \ ,$$

which are associative, etc. They obey our previous distributivity law.

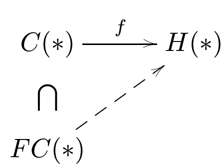
In these lectures it is the graded object which is what we mean when we say Hopf ring. We think of $H_*(Z \times BO)$ as a simple case of this concentrated in degree zero.

We give a complete description of $H_*\mathbf{H}_*$ as a Hopf ring. $\mathbf{H}_1$ is just the space $RP^\infty$ which we have already described. We denote $\beta_{(i)} = \beta_{2^i}$. For $I = (i_0, i_1, ...)$ a finite sequence of non-negative integers, define $\beta^I = \beta_{(0)}^{\circ i_0} \circ \beta_{(1)}^{\circ i_1} \circ \cdots$, and $\ell(I) = \Sigma i_k$. Then $H_*\mathbf{H}_n$ is the exterior algebra on generators $\beta^I$, $\ell(I) = n$. The coalgebra structure follows from $H_*\mathbf{H}_1$.

This has another, more appealing, description. $H_*\mathbf{H}_*$ is the "free" Hopf ring over the Hopf algebra, $H_*\mathbf{H}_1$. Of course we must ask; what is a "free" Hopf ring. For that matter, what is a free anything? If we have a set $S$ and we want to construct the free abelian group on $S$, $F(S)$, it has the property that any set map $f : S \to A$ to an abelian group factors uniquely through a canonical inclusion in $F(S)$.




Likewise, for a graded collection of Hopf algebras, $C(*)$, *the free Hopf ring on* $C(*)$, $FC(*)$, is a Hopf ring with a map of Hopf algebras $C(*) \to FC(*)$ such that any map of $C(*)$ into a Hopf ring factors uniquely through $FC(*)$:




It is a very elegant "global" description of $H_*\mathbf{H}_*$ to say it is the free Hopf ring on $H_*\mathbf{H}_1$. Although the idea of a free Hopf ring has been around for some time, it has only recently been rigorously defined, [Goe99] [HT98].

Needless to say, we could do the entire discussion for $H_*(X, \mathbb{F})$, the cohomology of $X$ with coefficients in a field $\mathbb{F}$. Replacing $\mathbf{H}_n$ we have the Eilenberg–MacLane spaces $K(\mathbb{F}, n)$. We apply $H_*(-;\mathbb{F})$ to these spaces to obtain a Hopf ring. Actually the first place $\mathbb{F}$ occurs we only need a ring $R$. Then $H_*(K(R,*),\mathbb{F})$ is a Hopf ring. We need $\mathbb{F}$ in the homology in order to insure a coalgebra.

We have just seen how quickly and easily our first example generalizes to different coefficients. We can now generalize this even further. Our generalization will include many known examples, whose computation and description depended heavily on the concept of Hopf rings. Later we will look at some techniques for computing Hopf rings.

Homology with coefficients, $H_*(-;G)$, satisfies a certain set of axioms, [ES52]. One of these simply states that the homology of a point is $G$. The cohomology theory $H^*(-;G)$ is classified by Eilenberg–MacLane spaces $K(G,n)$, i.e., $H^n(X;G) \cong [X, K(G,n)]$. We have $\Omega K(G,n+1) \cong K(G,n)$. Also, the homology can be defined using these spaces. We have maps $\Sigma K(G,n) \to K(G,n+1)$.

$$H_n X = \lim_{i\to\infty} [S^{i+n}, K(G,i) \wedge X] \ .$$

If we weaken the axioms slightly by eliminating the "axiom of a point" then we obtain generalized homology and cohomology theories which have many of the same formal properties of ordinary homology and cohomology [Whi62] [Ada69]. In particular, a (generalized) cohomology theory, $E^*(X)$, is a collection of abelian groups $\{E^n(X)\}$. We always assume we are working with ring theories so we have $E^*(X)$ is a graded ring. Brown's theorem tells us that there is a collection of spaces, $\mathbf{E}_* = \{\mathbf{E}_n\}$, such that $E^*(X) = [X, \mathbf{E}_*]$, i.e., $E^n(X) = [X, \mathbf{E}_n]$. The axioms still give a suspension isomorphism and $\Omega \mathbf{E}_{n+1} \cong \mathbf{E}_n$ follows as above. The generalized homology is given by $E_n(X) = \lim_{i\to\infty}[S^{n+i}, \mathbf{E}_i \wedge X]$. The collection, $\mathbf{E}_* = \{\mathbf{E}_n\}$, with the property $\Omega \mathbf{E}_{n+1} = \mathbf{E}_n$ is called an $\Omega$-spectrum. Any $\Omega$-spectrum gives us a cohomology (and homology) theory and vice versa, so the study of cohomology theories is equivalent to the study of $\Omega$-spectra. In particular, if you have a generalized cohomology theory you wish to study, all information you can find about its $\Omega$-spectrum should be useful in the long run. In our case, since we assume $E^*(X)$ is a ring, not only is each $\mathbf{E}_n$ an abelian group in the homotopy category, but $\mathbf{E}_*$ is a graded ring object in the homotopy category. So, if we apply homology with field coefficients to $\mathbf{E}_*$ we have a Hopf ring! However, we are trying to generalize so that is not enough. Let $G^*(X)$ be our cohomology theory classified by $\mathbf{G}_*$, and let $E_*(X)$ be our homology theory. Let us look at $E_*\mathbf{G}_*$. It is clear that $E_*\mathbf{G}_n$ is an algebra and we have maps

$$E_*\mathbf{G}_i \otimes E_*\mathbf{G}_j \to E_*\mathbf{G}_{i+j} \ ,$$

but in order to have our rich structure, the Hopf ring, we must have each $E_*\mathbf{G}_n$ be a coalgebra. We have maps

$$E_*\mathbf{G}_n \to E_*(\mathbf{G}_n \times \mathbf{G}_n) \leftarrow E_*\mathbf{G}_n \otimes E_*\mathbf{G}_n \ .$$

If the one on the right is an isomorphism we say we have a Künneth isomorphism and then $E_*\mathbf{G}_n$ is a coalgebra and $E_*\mathbf{G}_*$ is a Hopf ring. Honesty compels me to admit that Künneth isomorphisms seldom exist in this general setting. However, with special cases or conditions on $E$ and/or $G$, this does occur. Shortly I will be giving several examples. The simplest is, of course, the case where $E_*(-\ ) = H_*(-\ ;\mathbb{F})$. We always have our Künneth isomorphism in this case. At any rate, in our general discussions, we assume we have a Hopf ring.

There is a collection of generalized homology theories called Morava K-theories [Wür91]. For each odd prime, $p$, and each $n > 0$, there is a theory $K(n)_*(-)$. The coefficient ring, i.e., the Morava $K$-theory of a point, is $K(n)_* \cong Z/p[v_n, v_n^{-1}]$, with the degree of $v_n$ equal to $2(p^n - 1)$. It is rather difficult to give an elementary presentation of these theories explaining where they originate and how they fit into the scheme of things. Suffice it to say for now that they are intimately connected with complex cobordism which we describe soon. The reason we bring them up now is their property

$$K(n)_*(X \times Y) \cong K(n)_*X \otimes K(n)_*Y \ ,$$

which implies that $K(n)_*\mathbf{G}_*$ is always a Hopf ring. Several of the known examples of computed Hopf rings involve $K(n)_*(-)$. In particular, let $\mathbf{K}_* = K(Z/p, *)$. Then $K(n)_*\mathbf{K}_*$ is known [RW80], $H_*(\mathbf{K(n)}_*, Z/p)$ is also known as is $K(n)_*\mathbf{K(n)}_*$ [Wil84]. $E_*\mathbf{K(n)}_*$ is known for some more general $E_*(-)$ and $K(n)_*\mathbf{G}_*$ is known for some other special cases.

We pause now for a moment to discuss our future. We want to give a good example of a generalized homology theory and corresponding spectrum. Our example is complex cobordism. Then we will put a few restrictions on $E$ and $G$ and construct some elements and relations that always hold in a very general setting. For this we must introduce some formal groups. Then we will describe the Hopf rings when $\mathbf{G}_*$ is complex cobordism. After that we will describe some more special cases and then give some techniques for computations.

We move to our example, complex bordism. We want to define a sequence of abelian groups $\Omega_n(X)$. We use manifolds to do this [CF64]. Manifolds are a much better understood class of topological spaces than the general $X$ we wish to study. We use this understanding of manifolds to study the general $X$ and in the process find new information about the manifolds themselves.

We begin by considering all maps of all $n$-dimensional manifolds into $X$,

$$M^n \xrightarrow{f} X \ .$$

There are too many such manifolds and maps. So much like we did when we went to homotopy theory or when homology is defined using triangles, we put an equivalence relation on these maps. If we have another map, $g : N^n \to X$, we say $f$ and $g$ are equivalent, or bordant, if there is an $n+1$ dimensional manifold $W^{n+1}$ and map $F : W^{n+1} \to X$, such that the boundary, $\partial W^{n+1}$, of $W^{n+1}$ is the disjoint union of $M^n$ and $N^n$; and $F$ restricted to this boundary is the disjoint sum of $f$ and $g$. Let the equivalence classes be $\Omega_n X$. It is a finitely generated abelian group. All of the axioms for a generalized homology theory can be verified geometrically, or we could easily build a spectrum. Let $O_n$ be the $n$-th orthogonal group (again!) and $BO_n$ its classifying space. Take the Thom space of the universal bundle (the one point compactification of the total space of the bundle) to get $MO_n$. Our maps

$$\begin{array}{ccc} \xi_{n-1} \oplus R & \longrightarrow & \xi_n \\ \downarrow & & \downarrow \\ BO_{n-1} & \longrightarrow & BO_n \end{array}$$

give rise to

$$\Sigma MO_{n-1} \longrightarrow MO_n$$

and Thom transversality gives an isomorphism

$$\Omega_n X \cong \lim_{i \to \infty} [S^{n+i}, MO_i \wedge X] \ .$$

We usually denote $\Omega_n X$ by $MO_n X$ [Ati61]. A generalized cohomology theory can also be defined as

$$MO^n X \cong \lim_{i\to\infty} [\Sigma^{i-n} X, MO_i] \ .$$

This is called unoriented cobordism. As

$$[\Sigma^{i-n} X, MO_i] \cong [X, \Omega^{i-n} MO_i] \ ,$$

we define

$$\mathbf{MO}_n = \lim_{i\to\infty} \Omega^{i-n} MO_i \ .$$

This is the $\Omega$-spectrum giving unoriented bordism and cobordism. This is not so exciting because each $\mathbf{MO}_n$ is just a product of mod 2 Eilenberg–MacLane spaces [Tho54]. To get something more useful, all we need to do is put a little structure on the manifolds we use. In particular, we assume that the stable normal bundle has a complex structure, induced by a map

$$M^n \to BU \ ,$$

and that this structure restricts from $W^{n+1}$ to the boundary when we define bordism. We now get complex bordism, $MU_n X$ [Mil60] [Nov67]. Again, we have Thom spaces and

$$MU_n X = \lim_{i\to\infty} [S^{2i+n} X, MU_i \wedge X]$$

and

$$MU^n X = \lim_{i\to\infty} [\Sigma^{2i-n} X, MU_i] \ .$$

We let $\mathbf{MU}_n = \lim_{i\to\infty} \Omega^{2i-n} MU_i$ to get our $\Omega$-spectrum.

The spectrum $MU = \mathbf{MU}_* = \{\mathbf{MU}_n\}_{n\in Z}$ is well studied and we will give a good description of its Hopf ring. In particular, $H_*(\mathbf{MU}_*, Z)$ has no torsion [Wil73] and is a Hopf ring. Also $MU_*\mathbf{MU}_*$ can be computed and is a Hopf ring! Furthermore, many more $E_*\mathbf{MU}_*$ can be computed because of the special torsion free property of $H_*\mathbf{MU}_*$ [RW77] [Wil82].

Before we move on to general nonsense about $E_*\mathbf{G}_*$ we want to state a few facts about $MU$. The first is the coefficient ring, $MU_*$, the bordism of a point. It is a polynomial algebra on even dimensional generators [Mil60]

$$MU_* \cong Z[x_2, x_4, ...] \ .$$

A confusing but necessary fact is that $E_* = E^{-*}$ for all theories, so for complex cobordism its coefficient ring is a polynomial algebra on generators in the negative even degrees.

In the case of real $K$-theory we saw how important $RP^\infty$ was. Now in a complex theory the important space is $CP^\infty$. The complex cobordism of $CP^\infty$ is a power series ring, over the coefficient ring, on a two dimensional element, x [Ada74]:

$$MU^* CP^\infty \cong MU^*[[x]] \ .$$

This is dual to $MU_* CP^\infty$, which is free over $MU_*$ on generators $\beta_i \in MU_{2i} CP^\infty$. The coproduct is $\beta_n \to \Sigma_{i+j=n} \beta_i \otimes \beta_j$. The space $\mathbb{C}P^\infty$ has a product on it

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{m} \mathbb{C}P^\infty \ .$$

This turns $MU_*CP^\infty$ into a Hopf algebra. Recall that this is a group object and in fact it is an abelian group object. Dual to this we know $MU^*CP^\infty$ as an algebra, so all of the "group" information is contained in the power series

$$F(x_1, x_2) = m^*(x) = \Sigma a_{ij} x_1^i \hat{\otimes} x_2^j \in MU^*CP^\infty \hat{\otimes} MU^*CP^\infty \ ,$$

where $a_{ij} \in MU^{-2(i+j-1)}$. This is called a *formal group law*. The group properties force restrictions on the coefficients $a_{ij}$. For example, commutativity means that $a_{ij} = a_{ji}$. We can also see that $a_{10} = a_{01} = 1$ and the other $a_{n0}$ and $a_{0n}$ are zero. We say $G$ has a *complex orientation* if $G^*\mathbb{C}P^\infty$ and $G_*\mathbb{C}P^\infty$ have all of the same properties as $MU$ has for $\mathbb{C}P^\infty$. The only one necessary, which implies the rest, is that

$$G^*\mathbb{C}P^\infty \cong G^*[[x^G]] \ , \quad x^G \in G^2\mathbb{C}P^\infty \ .$$

If this is true, then we also have $\beta_i^G$ with the nice coproduct and distinguished elements $a_{ij}^G \in G^{-2(i+j-1)}$. We have our power series $F_G(x_1, x_2)$. We denote the formal group law by a formal group sum

$$F_G(x_1, x_2) = x_1 +_{F_G} x_2 \ .$$

The element $x \in G^2CP^\infty$ can be represented by a map $x^G \in [CP^\infty, \mathbf{G}_2]$. Assume also that $E$ has a complex orientation, we define $b_i$ by

$$x_*^G(\beta_i^E) = b_i^E \in E_{2i}\mathbf{G}_2 \ .$$

Of course, this may be zero, but we can still define it. We restrict our attention to $E$ and $G$ with complex orientation, however, our next construction does not depend on that.

While we do this, keep in mind the simple case of $H_*(Z)$ (Z the integers), that we have already discussed,

$$G^* = [\text{point}, \ \mathbf{G}_*] \ .$$

Thus $G^*$ is just the set of components given a graded ring structure by the ring structure of $\mathbf{G}_*$. For a $\in G^*$ we have a map $a : \text{pt} \to \mathbf{G}_*$ and we define the element $[a] \in E_0\mathbf{G}_*$ by $a_*(1) = [a], 1 \in E_0 = E_0(\textit{point})$.

Thus the "ring-ring" $E_*[G^*]$ is contained in the Hopf ring $E_*\mathbf{G}_*$. In particular we have elements $[a_{ij}^G] \in E_0\mathbf{G}_{-2(i+j-1)}$ and we can use the ring structure of the Hopf ring, with its $*$ for addition and $\circ$ for multiplication to define a new formal group law!

$$x +_{[F_G]} y = *_{i,j}[a_{ij}^G] \circ x^{\circ i} \circ y^{\circ j} \ .$$

We are nearly ready to state our main relation which relates the formal group laws for $E$ and $G$ to give unstable homotopy information. Let $b(s) = \Sigma b_i s^i \in E_*\mathbf{G}_2[[s]]$ as usual. Then

(Main relation) $\qquad b(s) +_{[F_G]} b(t) = b(s +_{F_E} t) \in E_*\mathbf{G}_*[[s,t]] \ .$

We are now ready to state the main theorems. In $E_*\mathbf{MU}_{2*}$, which is (an evenly graded) Hopf ring, we have $E_*[MU^*]$, and the $b_i$. We claim that $E_*\mathbf{MU}_{2*}$ is generated by these elements and the only relations are those given by the main relation. In particular this completely describes $MU_*\mathbf{MU}_{2*}$. To give all of $E_*\mathbf{MU}_*$ it is only necessary to add $e_1 \in E_1\mathbf{MU}_1$, with $e_1 * e_1 = 0$ and $e_1 \circ e_1 = \pm b_1$. (You get the minus sign if you read [BJW95] and the plus from [Goe99], the votes have not yet been counted.)

Thus we see that the "main relation" contains a lot of information about complex cobordism. It is, however, easy to prove, so we do it here. We just write down the maps

$$CP^\infty \times CP^\infty \xrightarrow{m} CP^\infty \xrightarrow{x^G} \mathbf{G}_2 \ .$$

We apply $E_*(-)$ and evaluate the image of $\beta(s) \otimes \beta(t)$ in $E_*(\mathbf{G}_2)$. By duality it is fairly easy to show that $m_*(\beta(s) \otimes \beta(t)) = \beta(s +_{F_E} t)$ in $E_*CP^\infty$. Thus, the notation which worked so well for us with $H_*(RP^\infty)$ is working even better here, where it would be impossible to write down the coefficients precisely. Apply $x^G_* \beta(s +_{F_E} t)$ to get $b(s +_{F_E} t)$. Our second evaluation thinks of $x^G$ as an element of $G^2CP^\infty$. Apply $m^*$ to $x^G$ to get $x_1 +_{F_G} x_2$ in $G^2(CP^\infty \times CP^\infty)$. Apply this element $(x_1 +_{F_G} x_2)_*$ to $\beta(s) \otimes \beta(t)$ to get $(x_1)_*\beta(s) +_{[F_G]} (x_2)_*\beta(t) = b(s) +_{[F_G]} b(t)$. This follows because the $+$ and $\times$ in $G^*(-)$ go to $*$ and $\circ$ respectively in $E_*\mathbf{G}_*$.

The main relation is easy to prove but the theorem that says that it gives all relations is very hard [RW77].

Let us look at some more examples of Hopf rings in this setting. Then we will describe an approach to computing this type of example. We have had one example of this type already, $H_*\mathbf{H}_*$, the mod 2 homology of the mod 2 Eilenberg-MacLane spectrum. Let $p$ be an odd prime and we will consider the Hopf ring $H_*(K(Z/p,*); Z/p)$, which we will also denote by $H_*\mathbf{H}_*$, letting $\mathbf{H}_n = K(Z/p, n)$ and $H_*(-)$ be the mod $p$ homology. Now $\mathbf{H}_1 = BZ/p$ with well known homology. Each $H_i\mathbf{H}_1$ is a $Z/p$ with generator $e_1 \in H_1\mathbf{H}_1$ and $a_i \in H_{2i}\mathbf{H}_1$. The element $e_1$ is an exterior generator and the $a_i$ give a divided power Hopf algebra. We let $a_{(i)} = a_{p^i}$. We use the standard map

$$CP^\infty \longrightarrow \mathbf{H}_2$$

which gives an inclusion $H_*\mathbb{C}P^\infty \hookrightarrow H_*\mathbf{H}_2$. This defines elements $b_i \in H_{2i}\mathbf{H}_2$. The "main relation" can be applied to this case but it tells us only that this image is a divided power algebra which we already know. Again define $b_{(i)} = b_{p^i}$.

One way to describe $H_*\mathbf{H}_*$ is as a free Hopf ring on $H_*\mathbf{H}_1$ and $H_*CP^\infty \hookrightarrow H_*\mathbf{H}_2$ with the relation $e_1 \circ e_1 = \pm b_1$. Signs enter in seriously in this Hopf ring. What do we mean? With a graded algebra, the concept of commutativity incorporates some signs. If $x$ has degree $i$ and $y$ has degree $j$, then $xy = (-1)^{ij}yx$. Now if $x$ has odd degree, then $x^2 = -x^2$. At an odd prime, as we are now, this implies that $x^2 = 0$. This is a very powerful statement. For a (graded) Hopf ring we must have a corresponding sign convention. We have a "minus one", [-1], which is just the abelian group "inverse map" or, in Hopf algebra language, a conjugation. If $HR_*(*)$ is a Hopf ring, with each $HR_*(n)$ a Hopf algebra, then for $x \in HR_i(k)$, $y \in HR_j(n)$, then

$$x \circ y = (-1)^{ij}[-1]^{\circ kn} \circ y \circ x \ .$$

This applies to our case. For $a_{(i)}$ and $a_{(j)}$ we get

$$a_{(i)} \circ a_{(j)} = [-1] \circ a_{(j)} \circ a_{(i)} \ .$$

Computing $[-1] \circ a_{(j)}$ to be $-a_{(j)}$ we get $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$. If $i = j$, then $a_{(i)} \circ a_{(i)} = -a_{(i)} \circ a_{(i)}$ and this implies $a_{(i)} \circ a_{(i)} = 0$ . No such restrictions are placed on the $b$'s. It is fairly easy to write down the Hopf algebras, $H_*\mathbf{H}_n$, but this is done elsewhere and is not enlightening [Wil82].

Of more interest is the Morava $K$-theory of the Eilenberg–MacLane spaces. We have already mentioned the existence of Morava $K$-theories, $K(n)_*(-)$. We continue to use $\mathbf{H}_*$ for the mod $p$ Eilenberg–MacLane spaces. We have computed $K(n)_*\mathbf{H}_*$. It is the free Hopf ring on $K(n)_*\mathbf{H}_1$. Let us describe $K(n)_*\mathbf{H}_1$. There are elements $a_i \in K(n)_{2i}\mathbf{H}_1$, $i < p^n$ with the usual nice coproduct. Defining $a_{(i)} = a_{p^i}$, $i < n$, we have $a_{(i)}{*}^p = 0$, $i < n-1$, just like a divided power algebra, but $a_{(n-1)}{*}^p = v_n a_{(0)}$. The previous sign arguments apply here to give $a_{(i)} \circ a_{(i)} = 0$. All elements

$$a_{(0)}^{\circ\varepsilon_0} \circ a_{(1)}^{\circ\varepsilon_1} \circ \cdots \circ a_{n-1}^{\circ\varepsilon_{n-1}} \qquad \varepsilon_k = 0, 1 \ ,$$

are non-zero in $K(n)_*\mathbf{H}_{\varepsilon_0+\cdots+\varepsilon_{n-1}}$. But notice we get $K(n)_*\mathbf{H}_k = 0$, $k > n$ we can always compute the $p$-th powers by the use of Hopf rings. If $\varepsilon_{n-1} = 0$, then the $p$-th power is zero. If $\varepsilon_{n-1} = 1$, then we can use the distributive law to compute the $p$-th power precisely. In particular, $K(n)_*\mathbf{H}_n$ is generated by an element $x$ with $x^{*p} = \pm v_n x$. This fact and the fact that $K(n)_*\mathbf{H}_k \cong 0$ $(k > n)$ are the main ingredients (together with the Morava structure theorem for complex cobordism) in the original proof of the geometric conjecture of Conner and Floyd [RW80].

For our final example we study the spectrum for Morava $K$-theory. Let $\mathbf{K(n)}_*$ be that $\Omega$-spectrum. We can define elements very similar to those already defined. We can do this for general $E_*(-)$, but $E$ has some very special technical restrictions. However, $E$ can be either mod $p$ homology, $H_*(-)$, or $K(n)_*(-)$. There are our usual elements $a_{(i)} \in E_{2p^i}\mathbf{K(n)}_1$, $i < n$, $b_{(i)} \in E_{2p^i}\mathbf{K(n)}_2$ and $e_1 \in E_1\mathbf{K(n)}_1$. The $p$-th powers are computed as above with the additional fact that $b_{(i)}{*}^p = 0$. Sign considerations still give us $a_{(i)} \circ a_{(i)} = 0$. The $p$-th power of $a_{(n-1)}$ is computed explicitly. The main relation shows how $b_{(k)}^{\circ p^n}$ can be written with lower $\circ$ powers. In the end each Hopf algebra is described explicitly and these few elements generate the Hopf ring [Wil84].

The interesting case here is $K(n)_*\mathbf{K(n)}_*$. For complex cobordism the interesting case was $MU_*\mathbf{MU}_*$. These are dual to their respective $E^*\mathbf{E}_*$, which are the same as $[\mathbf{E}_*, \mathbf{E}_*]$; and these are the *unstable* $E^*(-)$ *operations*. There are strong applications of these unstable operations for the $MU$ case.

As promised, we now give a brief description of how to compute Hopf rings such as $E_*\mathbf{G}_*$. The standard inductive approach to $E_*\mathbf{G}_*$ just uses the bar spectral sequence for a loop space [RS65], we can use this because $\Omega\mathbf{G}_{k+1} \cong \mathbf{G}_k$. The spectral sequence $E^r(E_*\mathbf{G}_k) \Rightarrow E_*\mathbf{G}_{k+1}$ has $E^2 \cong \mathrm{Tor}^{E_*\mathbf{G}_k}(E_*, E_*)$. We are assuming that there is a Künneth isomorphism for these spaces so we have a Hopf ring. This is then a spectral sequence of Hopf algebras. It comes from the geometric base construction:

$$\mathbf{G}_{k+1} \cong B\mathbf{G}_k \cong \coprod_{n\geq 0} D^n \times \underbrace{\mathbf{G}_k \times \cdots \times \mathbf{G}_k}_{n\text{-copies}} \ /\text{relations.}$$

This is filtered by

$$F^s B\mathbf{G}_k \cong \coprod_{s\geq n\geq 0} D^n \times \underbrace{\mathbf{G}_k \times \cdots \times \mathbf{G}_k}_{n\text{-copies}} \ /\text{relations.}$$

The spectral sequence is just a spectral sequence of this filtered space. The quotient is

$$F^sB\mathbf{G}_k/F^{s-1}B\mathbf{G}_k \cong \Sigma^s \wedge \underbrace{\mathbf{G}_k \wedge \cdots \wedge \mathbf{G}_k}_{s\text{-copies}} \ ,$$

and $E^1 \cong \otimes^s E_*\mathbf{G}_k$.

We can introduce Hopf rings [TW80] into this special sequence in a very simple way that allows us to keep track of the new ($\circ$) product in an inductive way. We consider the product

$$\mathbf{G}_{k+1} \times \mathbf{G}_n \longrightarrow \mathbf{G}_{k+1+n}$$

as

$$B\mathbf{G}_k \times \mathbf{G}_n \longrightarrow B\mathbf{G}_{k+n} \ .$$

This respects filtration in that

$$F^sB\mathbf{G}_k \times \mathbf{G}_n \longrightarrow F^sB\mathbf{G}_{k+n} \ .$$

Immediately we get a pairing

$$\circ : E^r(E_*\mathbf{G}_k) \otimes E_*\mathbf{G}_n \longrightarrow E^r(E_*\mathbf{G}_{k+n})$$

and differentials respect it: $d^r(x \circ y) = d^r(x) \circ y$. This says quite a bit as it is, but we can really evaluate this product precisely, inductively, because the pairing on

$$\begin{array}{ccc} F^sB\mathbf{G}_k/F^{s-1}B\mathbf{G}_k \times \mathbf{G}_n & \longrightarrow & F^sB\mathbf{G}_{k+n}/F^{s-1}B\mathbf{G}_{k+n} \\ \updownarrow \cong & & \updownarrow \cong \\ \Sigma^s \wedge \mathbf{G}_k \wedge \cdots \wedge \mathbf{G}_k \times \mathbf{G}_n & \longrightarrow & \Sigma^s \wedge \mathbf{G}_{k+n} \wedge \cdots \wedge \mathbf{G}_{k+n} \end{array}$$

is given by the map $\mathbf{G}_k \times \mathbf{G}_n \to \mathbf{G}_{k+n}$. On the $E^1$ term this means that the $\circ$ product can be evaluated by

$$(y_1 \otimes \cdots \otimes y_s) \circ x = \Sigma y_1 \circ x^{(1)} \otimes y_2 \circ x^{(2)} \otimes \cdots \otimes y_s \circ x^{(s)}$$

where $x \to \Sigma x^{(1)} \otimes \cdots \otimes x^{(s)}$ is the iterated reduced coproduct.

This spectral sequence pairing has been the main tool in most Hopf ring calculations we have done. Of course you have to know something to begin. If we are computing the mod $p$ homology, $H_*(-)$, then we know $H_0(\mathbf{G}_*)$ because it is just the ring-ring $Z/p[G^*]$. Then the spectral sequence can be used to compute by induction on degrees. It helps to know the *stable homotopy*,

$$H_nG \cong \lim_{i\to\infty} H_{i+n}(\mathbf{G}_i) \ .$$

Since this is unchanged once $n < i$. Knowing $H_0\mathbf{G}_*$ meant knowing $G^*$ which is just the stable homotopy

$$G^{-n} \cong G_n \cong \lim_{i\to\infty} [S^{i+n}, \mathbf{G}_i] \ .$$

So this technique we call *trapping* because, in essence, we trap the homology of $\mathbf{G}_*$ between the stable homotopy and homology!

If we want to compute $E_*\mathbf{G}_*$, it is sometimes convenient to compute the homology first and then use the Atiyah-Hirzebruch spectral sequence to compute $E_*\mathbf{G}_*$. In other cases, such as $K(n)_*\mathbf{H}_*$, $\mathbf{H}_0 \cong Z/p$, so $K(n)_*\mathbf{H}_0$ is just $K(n)_*[Z/p]$ and we can do the induction by spaces.

The technique seems quite powerful and many more application await the mathematician who likes to compute such things.

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