Mod-two cohomology of symmetric and alternating groups

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Joint work with Chad Giusti and Paolo Salvatore

Group cohomology

Recall that there is a unique homotopy type, by abuse any representative called BG, obtained by quotienting any free contractible G-space by its G-action.
It serves as a bridge between topology and algebra. For finite G,

$$H^*(BG; k) \cong \mathrm{Ext}_{k[G]}(k; k). \quad \text{(Almost by definition)}$$

This is the ground ring for Borel equivariant cohomology and "controls" the cohomology of X/G for any free G action.

$$K^0(BG) \cong \mathrm{Rep}_{\mathbb{C}}(G)_{\hat{I}}. \quad \text{(Atiyah-Segal)}$$

Despite being basic, relatively few explicit calculations of group cohomology rings have been made, a large percentage being in Adem & Milgram's book.

Hopf ring structure

Definition

A Hopf ring is a ring object in the category of coalgebras. Explicitly, a Hopf ring is vector space V with two multiplications, one comultiplication, and an antipode $(\odot, \cdot, \Delta, S)$ such that the first multiplication forms a Hopf algebra with the comultiplication and antipode, the second multiplication forms a bialgebra with the comultiplication, and these structures satisfy the distributivity relation

$$\alpha \cdot (\beta \odot \gamma) = \sum_{\Delta\alpha = \sum a' \otimes a''} (a' \cdot \beta) \odot (a'' \cdot \gamma).$$

Hopf ring structure

A free "algebra" on two products would be very large, but the distributivity relation cuts things down considerably.

$$\alpha \cdot (\beta \odot \gamma) = \sum_{\Delta\alpha = \sum a' \otimes a''} (a' \cdot \beta) \odot (a'' \cdot \gamma).$$

Consequence: every element can be reduced to Hopf monomials $m_1 \odot m_2 \odot \cdots \odot m_i$, where the m_i are monomials in the $\cdot$-product alone.

Mod-two chomology of symmetric groups

Theorem (Giusti-Salvatore-S)

$\bigoplus_{n\geq 0} H^*(B\mathcal{S}_{2n};\mathbb{F}_2)$ is a Hopf ring.
As such, it is generated by classes $\gamma_{\ell,m} \in H^{m(2^\ell-1)}(B\mathcal{S}_{m2^\ell})$, with $\ell, m \geq 0$, where $\gamma_{0,0} = 1_0$, the unit for transfer product, and $\gamma_{0,m} = 1_m$ the unit for cup product on component $2m$.
The coproduct of $\gamma_{\ell,m}$ is given by

$$\Delta\gamma_{\ell,m} = \sum_{i+j=m} \gamma_{\ell,i} \otimes \gamma_{\ell,j}.$$

Relations between transfer products of these are given by

$$\gamma_{\ell,n} \odot \gamma_{\ell,m} = \binom{n+m}{n} \gamma_{\ell,n+m}.$$

Cup products of generators on different components are zero, and there are no other relations between cup products of generators.

Mod-two chomology of symmetric groups - exercises

Find additive bases for cohomology of small symmetric groups.

Calculate products such as

- $(\gamma_{1,1} \odot 1_1) \cdot \gamma_{1,2}^2$
- $(\gamma_{1,1} \odot 1_1) \cdot (\gamma_{1,1}^2 \odot \gamma_{1,1})$
- $(\gamma_{1,1} \odot 1_1) \cdot \gamma_{2,1}$
- $(\gamma_{1,1}^3 \odot \gamma_{2,1} \odot 1_2) \cdot (\gamma_{1,2} \odot 1_4)$.

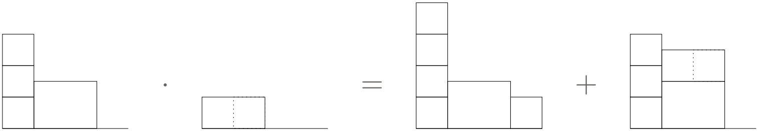
Or ponder this presentation as you like.

Talking with your neighbors encouraged.

Mod-two chomology of symmetric groups - solution to the last exercise

This Hopf ring has a diagrammatic presentation - which we call that of "skyline diagrams" - where the two products operate in two dimensions. The class $\gamma_{k,n}$ is represented by a box of width $2^{k-1}n$

and area equal to its degree (namely $(2^{k-1}-1)n$).



$$(\gamma_{1,1}^3 \odot \gamma_{2,1} \odot 1_2) \cdot (\gamma_{1,2} \odot 1_4) = \gamma_{1,1}^4 \odot \gamma_{2,1} \odot 1_2 + \gamma_{1,1}^3 \odot \gamma_{1,2}\gamma_{2,1} \odot 1_2$$

Mod-two chomology of symmetric groups - previous results

Theorem 5.4

$$H^*(S_{12}) \cong P[\sigma_1, \sigma_2, \sigma_3, c_3, \sigma_4, \sigma_5, \sigma_6, d_6, d_7, d_9](x_5, x_7, x_8)/\langle R\rangle$$

where deg $\sigma_i = i$, deg $c_3 = 3$, deg $d_i = i$, deg $x_i = i$ *and* R *is the set of relations*

$$x_7^2 + \sigma_4 x_5^2 + \sigma_2\sigma_4 x_8 + (\sigma_6 c_3 + \sigma_2\sigma_4 c_3) x_5 + \sigma_4^2 + c_3^2 + \sigma_2^2\sigma_4 d_6 + \sigma_6\sigma_2 d_6,$$
$$x_8^2 + d_6 x_5^2 + c_3 d_6 x_7 + (d_9\sigma_2 + c_3 d_6 \sigma_2) x_5 + d_6^2\sigma_2^2 + c_3^2\sigma_4 d_6 + d_9 c_3 \sigma_2,$$
$$x_5 x_7 + \sigma_2 x_5^2 + [\sigma_2^2 + \sigma_4] x_8 + \sigma_2^2 c_3 x_5 + \sigma_2^3 d_6 + \sigma_6 c_3^2 + \sigma_6 d_6 + \sigma_2\sigma_4 c_3^2,$$
$$x_5 x_8 + c_3 x_5^2 + [c_3^2 + d_6] x_7 + c_3^2 \sigma_2 x_5 + c_3^3 \sigma_4 + d_9 \sigma_2^2 + d_9\sigma_4 + c_3 d_6 \sigma_2^2,$$
$$x_5^3 + c_3\sigma_2 x_5^2 + x_7 x_8 + c_3\sigma_2^2 x_8 + \sigma_2 c_3^2 x_7 + (\sigma_2^2 d_6 + \sigma_4 c_3^2) x_5 + \sigma_2^3 d_9 + c_3^3\sigma_6 + \sigma_6 d_9,$$

$$d_9\sigma_1, d_9\sigma_3, d_9\sigma_5$$
$$d_7\sigma_3, d_7 x_5, d_7\sigma_1 c_3, d_7(x_7 + \sigma_4 c_3), d_7(\sigma_5 + \sigma_4\sigma_1), d_7(\sigma_6 + \sigma_4\sigma_2),$$
$$d_7(x_8 + d_6\sigma_2), d_7(d_9 + d_6 c_3),$$

$$x_7\sigma_1 + x_5(\sigma_1\sigma_2 + \sigma_3) + c_3(\sigma_2\sigma_3 + \sigma_2^2\sigma_1 + \sigma_1\sigma_4 + \sigma_5),$$
$$x_7\sigma_3 + x_5(\sigma_5 + \sigma_1\sigma_4) + c_3(\sigma_1\sigma_6 + \sigma_3\sigma_4 + \sigma_1\sigma_2\sigma_4),$$
$$x_7\sigma_5 + x_5\sigma_1\sigma_6 + c_3(\sigma_3\sigma_6 + \sigma_1\sigma_2\sigma_6),$$
$$x_8\sigma_1 + d_6(\sigma_3 + \sigma_1\sigma_2),$$
$$x_8\sigma_3 + d_6(\sigma_5 + \sigma_1\sigma_4),$$
$$x_8\sigma_5 + d_6\sigma_1\sigma_6.$$

Plan from here

1. How and where almost Hopf rings associated to series of groups arise.

2. Their current use in the cohomology of alternating groups, along with other ingredients.
    2.1 Relationships with symmetric groups, as in the Gysin sequence.
    2.2 Fox-Neuwirth resolutions.
    2.3 Restriction to elementary abelian subgroups.

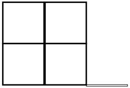
3. Further directions.

Another Hopf ring example

We may use Young diagrams to represent symmetric polynomials - that is elements of $R[x_1, \cdots, x_n]^{\mathcal{S}_n}$ - through symmetrized monomials.

$$P \leftrightarrow \mathrm{Sym}(x^P),$$

where $\mathrm{Sym}(m)$ denotes the minimal symmetrization of a monomial m, namely $\sum_{[\sigma] \in S_n/H} \sigma \cdot m$ where H fixes m.

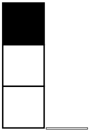


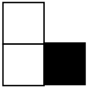
$$\leftrightarrow {x_1}^2{x_2}^2 + {x_1}^2{x_3}^2 + {x_2}^2{x_3}^2.$$

A diagrammatic approach to invariant theory

Multiplication is given by the sum of "stackings" over all possible column matchings up to automorphism.

This seems ineffective - one doesn't immediately see the fact that these are polynomial rings!
But doing this with colored blocks is a fruitful approach to invariants of multiple sets of variables.

 $\leftrightarrow {x_1}^2 y_1 + {x_2}^2 y_2.$

 $\leftrightarrow {x_1}^2 y_2 + {x_2}^2 y_1.$

A diagrammatic approach to invariant theory

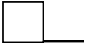
These diagrams immediately define additive bases for these rings of invariants, with multiplication still represented by stacking.

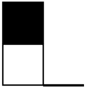
On the other hand, presenting these rings in positive characteristic by generators and relations over has been notoriously difficult.

In particular, Feshbach gave explicit generators and inductively defined relations for these rings over $\mathbb{F}_2$, and to our knowledge such descriptions are still open for these rings over $\mathbb{F}_p$ with $p > 2$.

A diagrammatic approach to invariant theory

A set of generators for $\mathbb{F}_2[x_1, x_2, y_1, y_2]^{\mathcal{S}_2}$ is

$a =$ , $b =$ , $c =$ , $d =$ 

$e =$ , $f =$ ,

With relations:

$$ab = e + f$$

and

$$ef = a^2 d + b^2 c.$$

Hopf ring structure

Direct sums of rings of symmetric invariants form Hopf rings.

The $\cdot$ multiplication is the standard one, defined to be zero if the number of variables does not agree.

The $\odot$ multiplication is an induction product: “reindex and symmetrize.”

The comultiplication is defined by “restricting” to get products of symmetric invariants of lower numbers of variables.

Hopf ring structure

Definition

A divided powers Hopf ring generated by a finite set $a_1, \cdots a_k$ is the Hopf ring generated under the two products by variables $a_{i,n}$ with $1 \leq i \leq k$ and $n \geq 1$ with coproducts determined by

$$\Delta a_{i,n} = \sum_{k+\ell=n} a_{i,k} \otimes a_{i,\ell},$$

and $\odot$-products

$$a_{i,n} \odot a_{i,m} = \binom{n+m}{n} a_{i,n+m}.$$

Hopf ring structure

Theorem (GSS)

The direct sum of rings of invariants in k collections of variables (for example, $\bigoplus_n A[x_1, \cdots, x_n, y_1, \cdots, y_n]^{\mathcal{S}_n}$ for $k = 2$) is a free divided powers $\mathbb{N}$-component Hopf ring, with k generators on the first component, given by the variables themselves.
The restriction maps to elementary abelian subgroups of symmetric groups respect Hopf ring structure and send generators to generators.

Hopf ring structure

There are similar Hopf ring structures on representation rings of symmetric groups, defined by tensor product, induction product, and restriction coproduct. The induction/restriction Hopf algebra structure was studied by Zelevinsky. (Similar games can be played for general linear groups over finite fields.)

Theorem (Strickland-Turner)

For any ring-theory E^*, the cohomology of symmetric groups $\bigoplus_n E^*(B\mathcal{S}_n)$ forms a (derived) Hopf ring where

- *The $\cdot$ product is the standard product (with zero products between distinct summands).*
- *The coproduct Δ is induced by the standard covering $p : B\mathcal{S}_n \times B\mathcal{S}_m \to B\mathcal{S}_{n+m}$.*
- *The product $\odot$ is the transfer associated to p.*

Hopf ring structures

Definition

A product series of finite groups is a collection $\{G_i\}_{i \geq 0}$ with embeddings $e_{n,m} : G_n \times G_m \hookrightarrow G_{n+m}$ which are associative and commutative up to conjugation.

Theorem (Giusti-S)

The (generalized) cohomology of a product series of groups forms a (derived) almost Hopf ring. For alternating groups, the deviation from being a Hopf ring is controlled by a polarization operator.

Hopf ring structures

Principle: natural ring-valued invariants of series of groups including the following tend to form almost Hpf rings.

- Group cohomology
- Representation rings
- Rings of invariants (of standard representations)
- Generalized cohomology (e.g. n-characters).

Almost Hopf ring? What breaks down?

Theorem (Giusti-S)

The (generalized) cohomology of a product series of groups forms a (derived) almost Hopf ring.
For alternating groups, the deviation from being a Hopf ring is controlled by a polarization operator.

The proof is "functorial" for all of the properties which define an almost Hopf ring.

Almost Hopf ring? What breaks down?

The proof is "functorial" for all of the properties which define an almost Hopf ring.

The one property not satisfied in general is that $(\odot, \Delta)$ form a bialgebra. In the case of symmetric groups, this does occur because

$$(\mathcal{S}_i \times \mathcal{S}_j)\backslash \mathcal{S}_n/(\mathcal{S}_k \times S_\ell) = \bigsqcup \mathcal{S}_n/(\mathcal{S}_p \times \mathcal{S}_q \times \mathcal{S}_r \times \mathcal{S}_s).$$

For other series of groups there is no such equality (though alternating groups are not far).
Ultimately this is good news: Hopf ring distributivity is the real "work horse," and this failure just requires more input of coproduct information.

Second ingredient for cohomology of alternating groups - the Gysin sequence

We have been using that if $H \subset G$ then there is both restriction and induction/ transfer in cohomology. For $\mathcal{A}_n \subset \mathcal{S}_n$ these fit into a short exact sequence

$$0 \to H^*(B\mathcal{S}_{2n})/e \xrightarrow{\text{res}} H^*(B\mathcal{A}_{2n}) \xrightarrow{\text{tr}} \mathrm{Ann}(e) \to 0,$$

where $\mathrm{Ann}(e)$ is the annihilator ideal of $e = \gamma_{1,1} \odot 1_{n-1}$.

Giusti and I compute this additively.

Third ingredient for cohomology of alternating groups - Fox-Neuwirth cochains

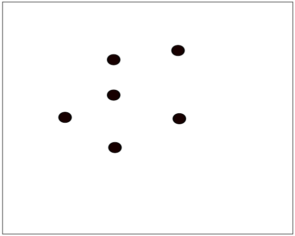
We denote by $\overline{\mathrm{Conf}}_n(\mathbb{R}^m)$ the quotient $\mathrm{Conf}_n(\mathbb{R}^m)/\mathcal{S}_n$, which is the space of unlabeled configurations of n points in $\mathbb{R}^m$. Recall that $\overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)$ is a model for $B\mathcal{S}_n$.

The cell structure is based on the dictionary ordering of points in $\mathbb{R}^m$ using standard coordinates, which we denote by $<$. This ordering gives rise to an ordering of points in a configuration.

Definition

Let the ith (depth of) agreement of a configuration be the number of consecutive coordinates shared by the ith and $i+1$st points (in the dictionary order) of a configuration.

Third ingredient for cohomology of alternating groups - Fox-Neuwirth cochains



For example, this configuration in the plane has agreements $[0, 1, 1, 0, 1]$.

Fox-Neuwirth cochains

Theorem (after Fox-Neuwirth)

For any list of $n-1$ non-negative integers $\lambda = [a_1, \ldots, a_{n-1}]$, the subspace of $Conf_n(\mathbb{R}^m)$ of configurations with λ as its list of agreements is homeomorphic to a Euclidean ball of dimension $mn - \sum a_i$. These subspaces are the interiors of cells in a CW structure on the one-point compactification $\overline{Conf}_n(\mathbb{R}^m)^+$.

By Alexander duality, the cellular chain complex computes the cohomology of $\overline{\mathrm{Conf}}_n(\mathbb{R}^m)^+$, which agrees through degree m with that of $\overline{\mathrm{Conf}}_n(\mathbb{R}^\infty) = B\mathcal{S}_n$.

Fox-Neuwirth cochains

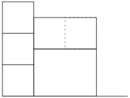
While the cells are immediate to enumerate the differential in the cellular chain complex, which is thus an alternative to the standard bar complex, as first given explicitly by Giusti-S, is complicated, involving signed counts of shuffles.

$$
\begin{aligned}
\delta([2,0,1,2]) &= -3[2,0,2,2] - 1[2,1,1,2] \\
\delta([1,0,2,2]) &= -1[1,1,2,2] + 1[1,2,2,1] + 2[2,0,2,2] - 1[2,2,1,1] \\
\delta([0,2,1,2]) &= -6[0,2,2,2] - 1[1,2,1,2] + 1[2,1,1,2] - 1[2,1,2,1] \\
\delta([0,1,2,2]) &= -4[0,2,2,2] - 1[1,2,2,1].
\end{aligned}
$$

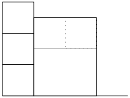
We also give a version "with orientations" for alternating groups.

Fox-Neuwirth cochains

Originally due to Fox-Neuwirth in two dimensions (braid groups) this cochain complex and related category was first extended by Vassiliev and then revisited by Batanin, Joyal, and Ayala-Hepworth in the context of n-categories.
Giusti-S give explicit cocyle representatives for $H^*(B\,\mathrm{sin}, \mathbb{F}_2)$ in the Fox-Neuwirth complexes.

Consider $\alpha =$ .
It is represented by the mod-two cochain BISym$[3, 0, 2, 1, 2, 0, 0]$, where BISym denotes the sum over all lists with the same set of consecutive collections of non-zero integers (so that for example $[0, 2, 1, 2, 0, 3, 0]$ would be another).

Digression: geometry of characteristic classes

$\alpha =$  defines a characteristic class in degree eight for an eight-sheeted covering map as follows for manifolds.

- Embed the covering $p : \widetilde{M} \to M$ in some $M \times \mathbb{R}^d \to M$.
- Consider the locus $\chi_\alpha(p) \subset M$ of m such that in $p^{-1}(m)$ there are two points which share their first three coordinates, and four points which share their first coordinate and can be partitioned into two groups of two which share their second coordinate.
- Then the characteristic class of p associated to α is Poincaré dual to $\chi_\alpha(p)$ (under standard transversality assumptions)

All characteristic classes of finite-sheeted covering spaces have similar descriptions in terms of agreement of coordinates in a fiber after embedding in a trivial Euclidean bundle.

Fourth ingredient for cohomology of alternating groups - restriction to elementary abelian subgroups

For $H \subset G$, the restriction $H^*(BG) \to H^*(BH)$ factors through $(H^*(BH))^{N(H)/H}$.

When H is elementary abelian, the resulting invariants are classical, for example including the Dickson invariants $\mathbb{F}_2[x_1, \cdots, x_n]^{GL_n(\mathbb{F}_2)}$.

We show that the cohomology of alternating groups is detected by elementary abelian subgroups, which by a theorem of Quillen implies no nilpotent elements.

Putting it all together

Giusti and I make full use of all of these techniques to give an almost-Hopf ring presentation of the mod-two cohomology of alternating groups.

Look on the handout and try to notice how it is similar to and different from the presentation for symmetric groups.

It is instructive to compare with computer calculations.

Further work - algebra

- Understand interplay of almost Hopf ring structures and other techniques, especially for modular representations.

- Connect to work on support varieties, etc.

- What is algebraic geometry of Hopf rings?

Further work - topology

- Integral cohomology (mod-p for $\mathcal{S}_n$ done by Lorenzo Guerra).
- Deepen understanding of Nishida and Adem relations (e.g. Fox-Neuwirth models of secondary relations?).
- Use Margolis homology and/or AHSS to approach Morava K-theory of $B\mathcal{S}_n$.
- Revisit Wellington's unstable Adams SS for QS^0 calculations.
- Revisit Galatius's characterization of the mod-two cohomology of stable mapping class group.