NAME ____________________

Astronomy 321

Final Examination

22 March 2017

Do 5 of the 7 following questions.

Constants:

$$
\begin{aligned}
\text{Solar Mass} &= M_\odot = 1.99 \times 10^{33}\ g \\
\text{Solar Radius} &= R_\odot = 6.96 \times 10^{10}\ cm \\
\text{Solar Luminosity} &= L_\odot = 3.9 \times 10^{33}\ erg\ s^{-1} \\
\text{Astronomical Unit} &= 1.49 \times 10^{13}\ cm \\
\text{Parsec} &= 3.08 \times 10^{18}\ cm \\
c &= 3.0 \times 10^{10}\ cm\ s^{-1} \\
e &= = 4.8 \times 10^{-10}\ e.s.u. \\
h &= 6.626 \times 10^{-27}\ erg\ s \\
m_p &= 1.67 \times 10^{-24}\ g \\
m_e &= 9.11 \times 10^{-28}\ g \\
k &= 1.38 \times 10^{-16}\ erg\ K^{-1} \\
\sigma &= 5.67 \times 10^{-5}\ erg\ cm^{-2}\ K^{-4}\ s^{-1} \\
G &= 6.67 \times 10^{-8}\ dyne\ g^{-2}\ cm^{2}
\end{aligned}
$$

Table of Selected Nuclide Masses

Element	Mass/(A×AMU)
H, Z=1,A=1	1.00782505
He,Z=2,A=4	1.0006508135
C, Z=6,A=12	1
N, Z=7,A=14	1.0002195718
O, Z=8,A=16	0.9996821637
Fe, Z=26,A=56	0.9988381696

Question 1:

A star like the Sun burns ~ 10 % of its hydrogen to helium while on the Main Sequence. The nuclear processing takes place in the core of the Sun.

a. Using the Virial Theorem, show that the central density ρ_c and pressure P_c of the core are related by $P_c = 0.36\ GM_c^{2/3}\rho_c^{4/3}$, with M_c the mass of the helium core.

b. Find an expression for the core temperature T_c using the P_c from Part a if the ideal gas law is applicable.

c. At the temperature at which helium ignites, $T_{ign} \sim 10^8$ K, what is the core density ρ_c based on your answer to (b)? How does this compare to the critical density above which electrons are degenerate in a gas with temperature $T_{ign} = 10^8$ K?

Question 2:

We investigate white dwarf cooling in this question. The first law of thermodynamics is

$$\mathrm{dQ} = \mathrm{dU} + \mathrm{dW} \tag{1}$$

where dQ is change in heat, dU is the change in internal energy and dW is the work. Assume that white dwarfs cool with constant radius, R_*, and that the internal temperature T can be approximated as $\mathrm{T} = T_{eff}$ where T_{eff} is related to L_* and R_* as $L_* = 4\pi R_*^2 \sigma T_{eff}^4$.

a. Show that the first law of thermodynamics can be rewritten as

$$-L_* dt = -C_V M_* dT_{eff} \tag{2}$$

where dt is the change in time, C_V is the specific heat at constant volume, $C_V = 1.5(k/m)$ where m is average mass of a particle in the gas, and M_* is the mass of the white dwarf.

b. If the white dwarf cools from luminosity $L_\circ$ to a final luminoisty of L_f, show that

$$t = -\frac{M_* C_V}{(4\pi\sigma R_*^2)^{1/4} L_f^{3/4}}(1 - (\frac{L_f}{L_i})^{3/4}) \tag{3}$$

c. Use the cooling time expression in Part b to estimate the time it takes for a 0.5 $M_\odot$ white dwarf, $R_* \sim 10^9$ cm, to cool from 0.01 $L_\odot$ to 0.0001 $L_\odot$.

d. What would be the consequence of observing a low luminosity, 0.5 $M_\odot$ white dwarf as described in Part c?

Question 3:

Assume the Milky Way galaxy contains $\sim 10^{11}$ stars that were formed with an initial mass function

$$\frac{\mathrm{dN}}{\mathrm{dM}} = a\mathrm{M}^{-2.35} \tag{4}$$

over the mass range 0.4 $\mathrm{M}_\odot$ to 100 $\mathrm{M}_\odot$.

a. Estimate the fraction of stars that form with mass $M_* > 8\ M_\odot$, the lower limit for a star to undergo a Type II supernova outburst.

b. Binary pulsar systems are composed of two rotating neutron stars in orbit about each other. Suppose that progenitor neutron star binaries are formed from random draws from the initial mass function. If half of all stars are in binary star systems, how many binaries formed in the galaxy from stars whose initial masses were $> 8\ \mathrm{M}_\odot$?

c. Suppose that progenitor black hole binary systems also form from random draws from the initial mass function. If half of all stars are in binary star systems, how many binaries formed in the galaxy from stars whose initial masses were $> 30\ \mathrm{M}_\odot$, the rough mass limit for formation of black holes from Type II supernovas?

d. Will a binary system composed of a 10 $\mathrm{M}_\odot$ star and 12 $\mathrm{M}_\odot$ star remain bound after the 12 $\mathrm{M}_\odot$ star undergoes a Type II supernova outburst? Assume that newly formed neutron star has mass $\mathrm{M}_* = 1.4\ \mathrm{M}_\odot$. What happens to the binary system after the 10 $\mathrm{M}_\odot$ star undergoes a Type II supernova outburst and leaves a neutron star? Do black hole binary progenitors suffer this same problem?

Question 4:

We used the equations of stellar structure, and Kramer's opacity to show that

$$L \propto M^{5.5} R^{-0.5} \tag{5}$$

earlier. The dependence on R was then eliminated through consideration of the nuclear burning process used to generate the star's energy. We consider a tweak of this result.

a. Find an approximate expression for the mean mass of a paricle in a plasma, in terms of the mass fractions X, Y, and Z, the (Z,A) of an element, and $m_\circ$ the atomic mass unit.

b. In the original derivation, the average mass per particle was assumed to be the same for all Main Sequence stars and so not included in the derivation of the mass-luminosity relation. Re-do the derivation of the mass-luminosity relation but this time retain the dependence on the composition of the gas, that is, retain the average mass of particle or retain the dependence on the mass fractions, X, Y, and Z.

c. Compare the luminosities for Pop I, Pop II, and Pop III stars of a given mass M_* using the mass-luminosity relation found in Part b. Pop III stars are the first generation of stars, $Z \sim 0$, Pop II stars are the next generation of stars, $Z \sim 0.001$, and Pop I stars are the cunrrent generations of stars $Z \sim$ 0.01-0.03.

Question 5:

Accelerated masses produce gravitational waves according to General Relativity. They lose energy in a manner analogous to electromganetic radiation with energy loss rate,

$$\frac{\mathrm{dE}}{\mathrm{dt}} = -\frac{32\mathrm{G}\Omega^6\mathrm{I}^2}{5\mathrm{c}^5} \tag{6}$$

where I is the quadrupole moment, G is the gravitational constant, Ω is the orbital frequency in Radians per second, and c is the speed of light. Assume that gravitational radiation is weak so that the orbital energy of the binary system is nearly constant over an orbit. Under this assumption, the orbits of the stars change adiabatically as they lose E and shrink in size.

a. For circular orbits and the assumption of an adiabatic change, show that $\dot{P}/P =$ -(3/2)$\dot{E}/E$ where P is the orbital period and E is the total energy. Under an adiabatic change, the orbits can be assumed to remain circular in shape evolving through a series of quasi-equilibrium states of decreasing orbital radius.

b. Find an expression for $\Omega(t)$ using the result in Part a and the energy loss rate given for gravitational radiation. Note that the quadrupole moment changes with time as the orbit shrinks.

c. For two black holes of mass 30 $\mathrm{M}_\odot$, make a plot of the evolution of $\Omega(t)$ up to the time when the separation distance is equal to 2 R_{sch} where R_{sch} is the Schwarschild radius for a spherical black hole of mass 30 $\mathrm{M}_\odot$.

Question 6:

a. Show using dimensional arguments that for electrons, quantum effects (degneracy) become important when the density exceeds

$$\rho \propto \mu_e m_h \left(m_e k T\right)^{1.5} \hbar^{-3}. \tag{7}$$

Here ρ is the mass density, μ_e is the mean moleular weight per electron $\mu_e = 2/(1+\mathrm{X})$, m_h is the the hydrogen mass, m_e is the electron mass, $\hbar$ is Planck's constant, k is the Boltzmann constant, and T is the gas temperature. Evaluating exactly leads to $\rho > 10.6 \times (T/10^6\mathrm{K})^{1.5}$ gram per cubic centimeter.

b. The *Helium Flash* begins when the density is $\sim 10^5$ g cm^{-3} and T $= 10^8$ Kelvin. Calculate the temperature when degeneracy is lifted.

c. Calculate the ratio of the energy generation rate at the time degeneracy is lifted to the energy generation rate at the onset of the flash.

Question 7:

A simple model for a star is to assume that the star is pure hydrogen and that the star has density structure given by

$$\rho(r) = \rho_\circ(1 - [\frac{r}{\mathrm{R}_*}])^n \tag{8}$$

where $\rho_\circ$ and n are constants. Here M_* is the stellar mass and R_* is the stellar radius.

a. For $n = 0$ and 2, plot $\mathrm{M}(r)$. Define $\rho_\circ$ so that the total mass is M_*.

b. For $n = 2$, find P(r). Find an expression for the central pressure of the star for this model.

c. Find an expression for the central temperature of the star if the star is dominated by gas pressure. Use the ideal gas law. Given $\mathrm{M}_* = \mathrm{M}_\odot = 2\times10^{33}$ g and $\mathrm{R}_* = \mathrm{R}_\odot = 7\times10^{10}$ cm evaluate your expression for T_c to find the central temperature of the star in Kelvin.)

d. Find the central temperature for the star if it is composed of helium rather than hydrogen. To ignite helium, what must be the radius of the star if its mass is unchanged?