White Dwarf Stars

A white dwarf is a [stellar core remnant](#) composed mostly of [electron-degenerate matter](#). A white dwarf is very [dense](#): its mass is comparable to the [Sun](#)'s, while its volume is comparable to the [Earth](#)'s. A white dwarf's faint [luminosity](#) comes from the [emission](#) of residual [thermal energy](#); no fusion takes place in a white dwarf. The nearest known white dwarf is [Sirius B](#), at 8.6 light years, the smaller component of the Sirius [binary star](#). There are currently thought to be eight white dwarfs among the hundred star systems nearest the Sun.The unusual faintness of white dwarfs was first recognized in 1910. The name white dwarf was coined by [Willem Luyten](#) in 1922.

Wikipedia entry

Computational Project

Goals:

- Write a numerical computational code to find white dwarf equilibria.
- Calculate white dwarf equilibrium structures.
- Find the mass-radius relationship for white dwarf stars using Newtonian physics.
- Find Chandrasekhar limit.
- Find the effects of a general relativistic calculation on the white dwarf mass radius relation. What happens to the Chandraskehar Limit?

Newtonian Structure Equations:

1. Mass Conservation

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

2. Hydrostatic Equilibrium

$$\frac{dP(r)}{dr} = -G\frac{M(r)}{r^2}\rho(r)$$

We have three unknown quantities, *P(r)*, *M(r)*, and ρ*(r)*, however, because for degenerate gases the density and pressure are related as *P = f(ρ)*. We can eliminate one variable and solve for two. We rewrite hydrostatic equilibrium as

$$\frac{dP(r)}{dr} = \frac{\partial P(\rho)}{\partial \rho}\frac{d\rho}{dr} \rightarrow \frac{d\rho}{dr} = -\left(\frac{\partial P(\rho)}{\partial \rho}\right)^{-1} G\frac{M(r)}{r^2}\rho(r)$$

We then solve for *M(r)* and *ρ(r)* using

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{d\rho}{dr} = -\left(\frac{\partial P(\rho)}{\partial \rho}\right)^{-1} G \frac{M(r)}{r^2} \rho(r)$$

The equation of state for degenerate electrons *P(ρ)*, is given on the next slide.

The degenerate electron pressure is

$$P(\rho) = \frac{\pi m_e^4 c^5}{3h^3} f(x) = 6.003 \times 10^{22} f(x) \ dynes/cm^2$$

where

$$f(x) = x\left(2x^2 - 3\right)\left(x^2 + 1\right)^{1/2} + 3 \ \sinh^{-1} x$$

and

$$x = \frac{h}{mc}\left(\frac{3}{8\pi} n_e\right)^{1/3} = 1.009 \times 10^{-2}\left(\frac{\rho}{\mu_e}\right)^{1/3}$$

For our calculation, we need

$$\frac{dP}{d\rho} = \frac{\partial P}{\partial x}\frac{dx}{d\rho} = 6.003 \times 10^{22}\left(\frac{8x^5}{3\rho\sqrt{1+x^2}}\right)$$

where

$$x = \frac{h}{mc}\left(\frac{3}{8\pi}n_e\right)^{1/3} = 1.009 \times 10^{-2}\left(\frac{\rho}{\mu_e}\right)^{1/3}$$

Try showing that in the limits x << 1 and x >> 1, the electron pressure goes to the appropriate non-relativistic and ultra-relativistic limits.

$$P(\rho) = \frac{\pi m_e^4 c^5}{3h^3} f(x) = 6.003 \times 10^{22} f(x) \ dynes/cm^2$$

$$\rightarrow P(\rho) = 10^{13} \left(\frac{\rho}{\mu_e}\right)^{5/3} and \ P(\rho) = 1.24 \times 10^{15} \left(\frac{\rho}{\mu_e}\right)^{4/3}$$

where the pressures are in *c.g.s, dynes-cm^{-2}*.

We solve for *M(r)* and *ρ(r)*, two unknown functions, and have two equations. This is good. To solve uniquely for our solution, however, we must impose two conditions on the solution.

1. If we specify less than two conditions, our solution is under-determined and we cannot uniquely specify a solution

2. If we specify more than two conditions, our solution is over-determined and we may not be able to solve our problem.

To solve for *M(r)* and *ρ(r)*, we may have two types of conditions. For example,

- Both conditions could apply at one end of our domain, the center or the surface.

1. One condition may apply at one end of the domain and the other at the other end of the domain, that is, one at the center and one at the surface.

The former is an ***Initial Value Problem*** and the latter is an ***Eigenval*ue Problem**.

To solve for *M(r)* and *ρ(r)*, (i) we impose a condition at the center of the star where we require that **M(0) = 0 at r = 0** and (ii) we require that **P goes to 0 at r = R$_*$**. Notice that we do not specify either the mass or radius of the star. The mass **M$_*$** and radius **R$_*$** are results of our calculation.

Our calculation, by its nature is a Boundary Value problem, an Eigenvalue Problem.

The method we choose to solve our equations does not seem consistent with these ideas. We choose to use what is referred to as a **Shooting Method**;

"In [numerical analysis](#), the shooting method is a method for solving a [boundary value problem](#) by reducing it to an [initial value problem](#). It involves finding solutions to the initial value problem for different initial conditions until one finds the solution that also satisfies the boundary conditions of the boundary value problem. In layman's terms, one "shoots" out trajectories in different directions from one boundary until one finds the trajectory that "hits" the other boundary condition"

Wikipedia entry

Shooting Method:

We start at the center of the star and integrate until we reach the surface. The idea will be to advance **_M(r)_** and **_ρ(r)_** from point to point as

$$M(r + \delta r) \approx M(r) + \delta r \frac{dM(r)}{dr} = M(r) + 4\pi r^2 \rho(r)\delta r$$

$$\rho(r + \delta r) \approx \rho(r) + \delta r \frac{d\rho}{dr} = \rho(r) - \delta r \left(\frac{\partial P(\rho)}{\partial \rho}\right)^{-1} G \frac{M(r)}{r^2} \rho(r)$$

Note that we use only first derivatives and the first power of δr, this a first order Euler scheme. (Compare to a Taylor series)

Shooting Method:

We have a problem though, at **_r = 0_**, we cannot begin our calculation,

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \to 0$$

and

$$\frac{d\rho}{dr} = -\left(\frac{\partial P(\rho)}{\partial \rho}\right)^{-1} G \frac{M(r)}{r^2} \rho(r)$$

is indeterminate

Shooting Method:

We start at $r = \Delta r$, where $\Delta r << R_*$ and

$$\frac{dM(r)}{dr} = \frac{d\Delta M}{dr} = 4\pi r^2 \left(\rho(0) + \Delta\rho \right) \rightarrow \Delta M \approx \left(\frac{4\pi}{3} \right) \Delta r^3 \rho(0)$$

This implies

$$-\left(\frac{\partial P(\rho)}{\partial \rho} \right)^{-1} G \frac{M(r)}{r^2} \rho(r) \rightarrow -\left(\frac{\partial P(\rho)}{\partial \rho} \right)^{-1} G \left(\frac{4\pi}{3} \right) \Delta r \rho(0)$$

Shooting Method:

(i) We move a little off-center and choose $\rho(0)$ and $r = \Delta r$. Using these values we can calculate the derivatives and start integrating.

(ii) We have a question, what we should use for our step size, δr? First, calculate some characteristic step lengths by finding

$$\delta r_M = \left| \frac{M(r)}{dM/dr} \right|, \quad \delta r_\rho = \left| \frac{\rho(r)}{d\rho/dr} \right|$$

Shooting Method:

(iii) What have we just done?

(iv) Choose the smaller of δr_M and δr_ρ and then say, let's allow only 1 % change in the variable over a step. In this way, we guarantee that our simple estimate *(low-order method*) is good.

(v) We use this method to re-calculate our step size on every step to keep our method accurate.

(vi) Halt the integration when ρ reaches a small value say $\rho/\rho(0) \leq 10^{-8}$ or so. This ends the star and determines M_* and R_*.

Tolman-Oppenheimer-Volkoff Equation

The **Tolman–Oppenheimer–Volkoff (TOV) equation** constrains the structure of a spherically symmetric body of isotropic material which is in static gravitational equilibrium, as modeled in general relativity. The **TOV equation** appropriate for white dwarfs is

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho\left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 P}{mc^2}\right)\left(1 - \frac{2Gm}{rc^2}\right)^{-1}$$
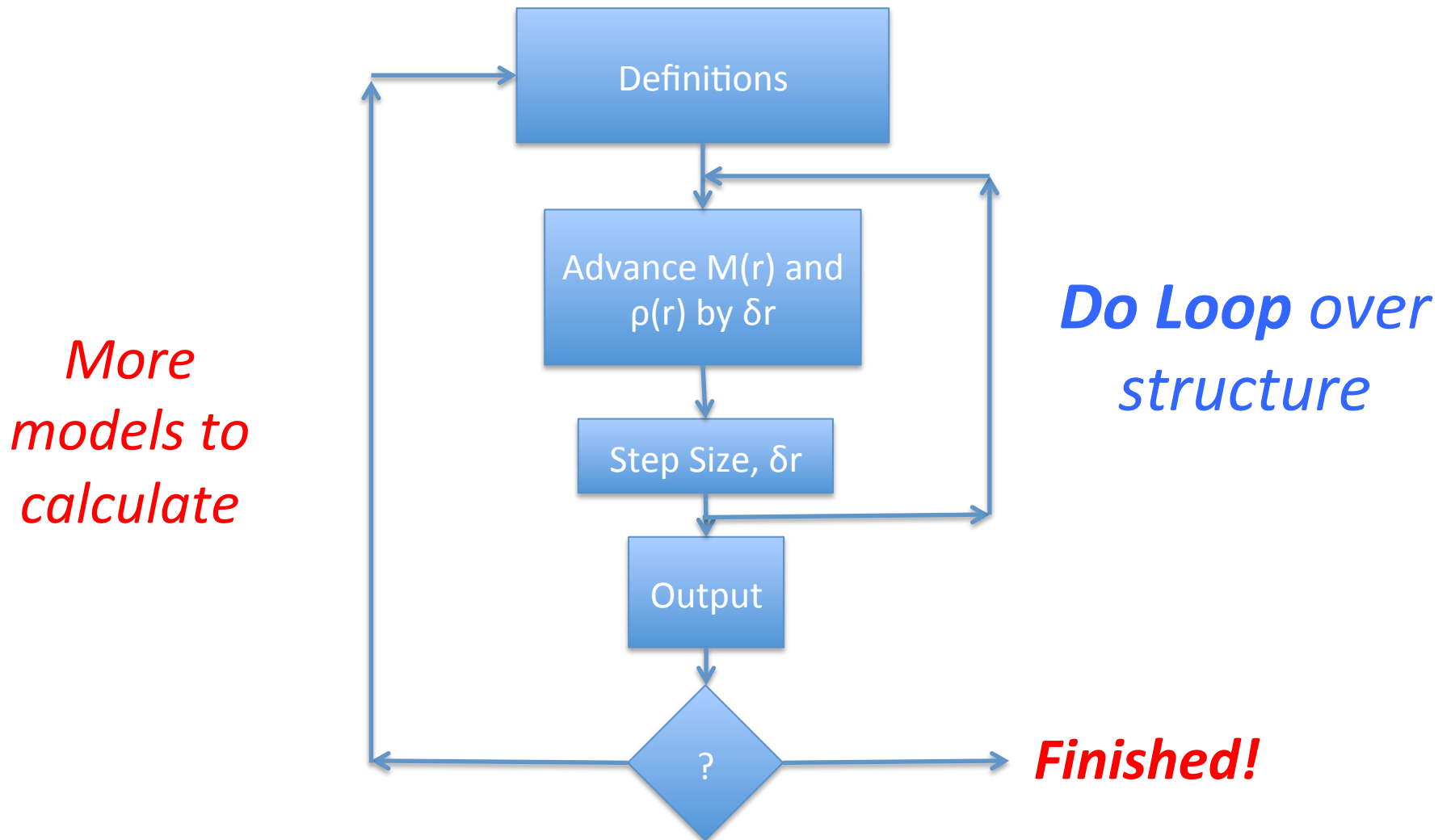
where, **m(r)** is the mass contained within radius **r** and **c** is the speed-of-light. TOV equation is similar in form to the Newtonian equation; it takes account of the fact that energies contribute to the gravitational mass.

Computational Project Report

1. Short introduction to problem and describe goal of your calculation.

2. Describe your numerical method, how you solved equation set, and how you tested your code. Turn in an annotated copy of your computer code.

3. Describe the results of your calculations and make sure to turn in a table that contains the ***central density***, R_* and M_* from your calculation and a plot of the mass-radius relation which includes the low-mass approximation.

4. Include effects of General Relativity and find the mass-radius relationship.

Appendix: Sample Code

Sample program (in Fortran 90). General flow of the program is

Definitions

Advance M(r) and ρ(r) by δr

Step Size, δr

Output

?

More models to calculate

Do Loop over structure

Finished!

```fortran
program mass_radius

xmue=2.00     ! pure helium or metals
pi=3.141592654

p1=5.0/3.0
p2=1.0/3.0
epsilon=2.0

do nwd=1,28

  rho0=5.0e4*epsilon**(nwd-1)
  r0=1.0e4
  xmass0=(4.0*pi/3.0)*r0**3*rho0
  rho=rho0
  r=r0
  xmass=xmass0
  step=r0
```

```fortran
   do n=1,20000
     x=1.009e-02*(rho/xmue)**p2
     dpdrho=6.002e22*8.0*x**5/(3.0*rho*sqrt(1.0+x**2))
     dm=4.0*pi*r**2*rho
     drho=-6.673e-08*xmass*rho/r**2/dpdrho
!  advance m, rho, and r
     xmass=xmass+step*dm
     rho=rho+step*drho
     r=r+step
     ts1=abs(rho/drho)
     ts2=abs(xmass/dm)
     step=0.01*amin1(ts1,ts2)
     if(rho.le.1.0e-00) go to 1
   end do
1 continue
```

```
  write(6,100) rho0,r,xmass/2.0e33,xn-1
100 format(1p4e12.4)

end do

end
```