Section 3: Simple Stellar Structure Models

We explore simple stellar models and some properties of Main Sequence stars. We do not solve the full set of coupled differential equations, rather, we find particular solutions for simplifying assumptions.

1. We make further dimensional arguments to see if we can understand more properties of Main Sequence stars.
2. We consider two prescriptions for the density, (i) constant density, (ii) linear fall off of density. The solutions allow us to see how well our simple scaling arguments perform.

Equations of Stellar Structure

I. $$\frac{dM(r)}{dr} = 4\pi r^2 \rho$$

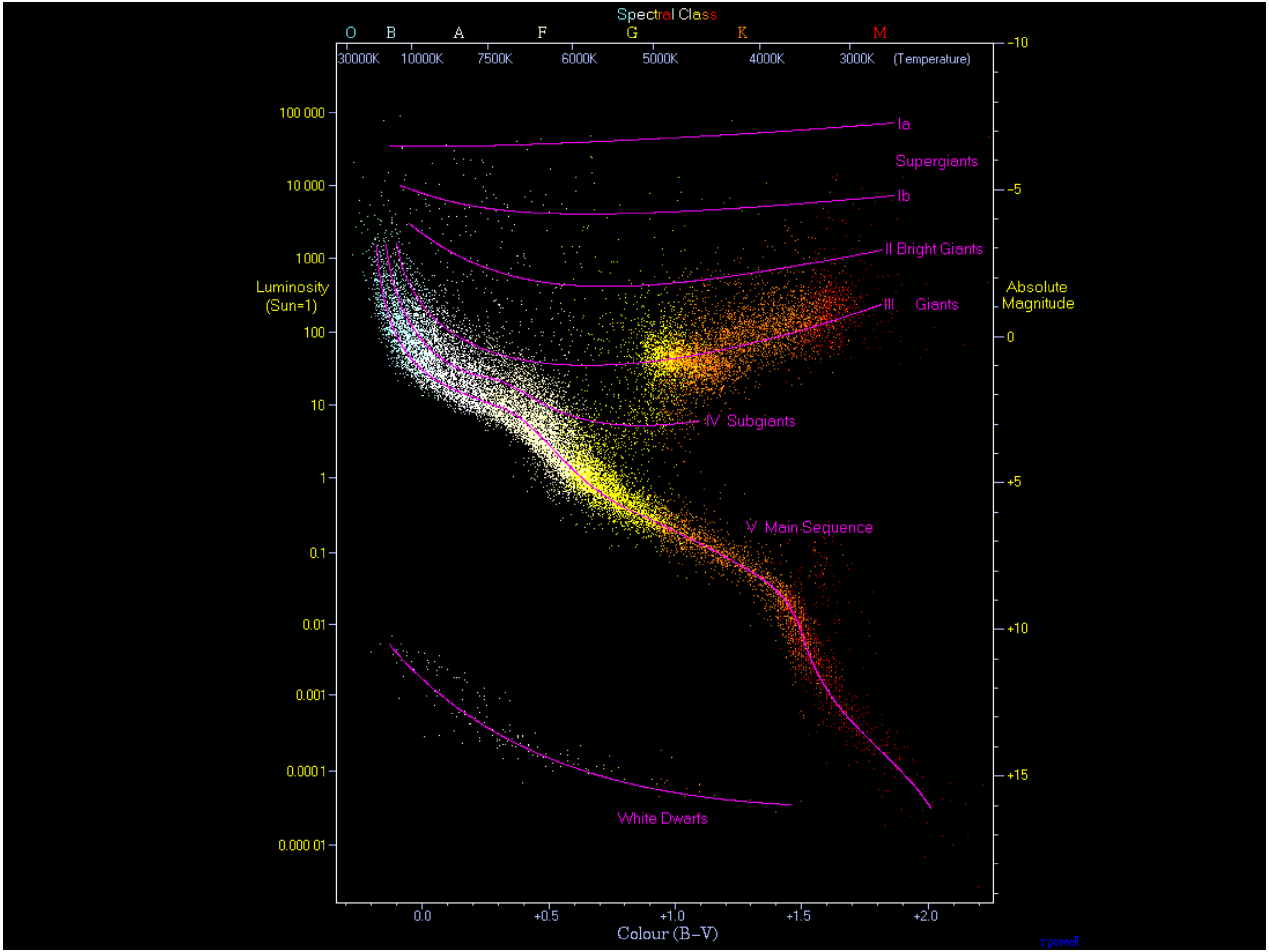
II. $$\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho$$

III. $$\frac{dL(r)}{dr} = A(r)\rho(r)\varepsilon(r), \;\; L = -\frac{16\pi r^2 acT^3}{3\kappa\rho}\frac{dT}{dr}$$

Dimensional Analysis

We first see if we can understand further properties of Main Sequence stars using dimensional analysis. We look at

- Mass-Luminosity relationship
- Upper and lower mass limits on Main Sequence stars
- Mass-radius relationship



Spectral Class
O
B
A
F
G
K
M
30000K
10000K
7500K
6000K
5000K
4000K
3000K
(Temperature)
100 000
10 000
1000
Luminosity
(Sun=1)
100
10
1
0.1
0.01
0.001
0.0001
0.000 01
-10
-5
Absolute
Magnitude
0
+5
+10
+15
Ia
Supergiants
Ib
II Bright Giants
III Giants
IV Subgiants
V Main Sequence
White Dwarfs
0.0
+0.5
+1.0
+1.5
+2.0
Colour (B–V)
r.powell

Table of main-sequence stellar parameters[25]

Stellar Class	Radius $R/R_\odot$	Mass $M/M_\odot$	Luminosity $L/L_\odot$	Temp. K	Examples[26]
O6	18	40	500,000	38,000	Theta[1] Orionis C
B0	7.4	18	20,000	30,000	Phi[1] Orionis
B5	3.8	6.5	800	16,400	Pi Andromedae A
A0	2.5	3.2	80	10,800	Alpha Coronae Borealis A
A5	1.7	2.1	20	8,620	Beta Pictoris
F0	1.3	1.7	6	7,240	Gamma Virginis
F5	1.2	1.3	2.5	6,540	Eta Arietis
G0	1.05	1.10	1.26	5,920	Beta Comae Berenices
G2	1	1	1	5,780	**Sun**[note 2]
G5	0.93	0.93	0.79	5,610	Alpha Mensae
K0	0.85	0.78	0.40	5,240	70 Ophiuchi A
K5	0.74	0.69	0.16	4,410	61 Cygni A[27]
M0	0.51	0.60	0.072	3,800	Lacaille 8760
M5	0.32	0.21	0.0079	3,120	EZ Aquarii A
M8	0.13	0.10	0.0008	2,660	Van Biesbroeck's star[28]

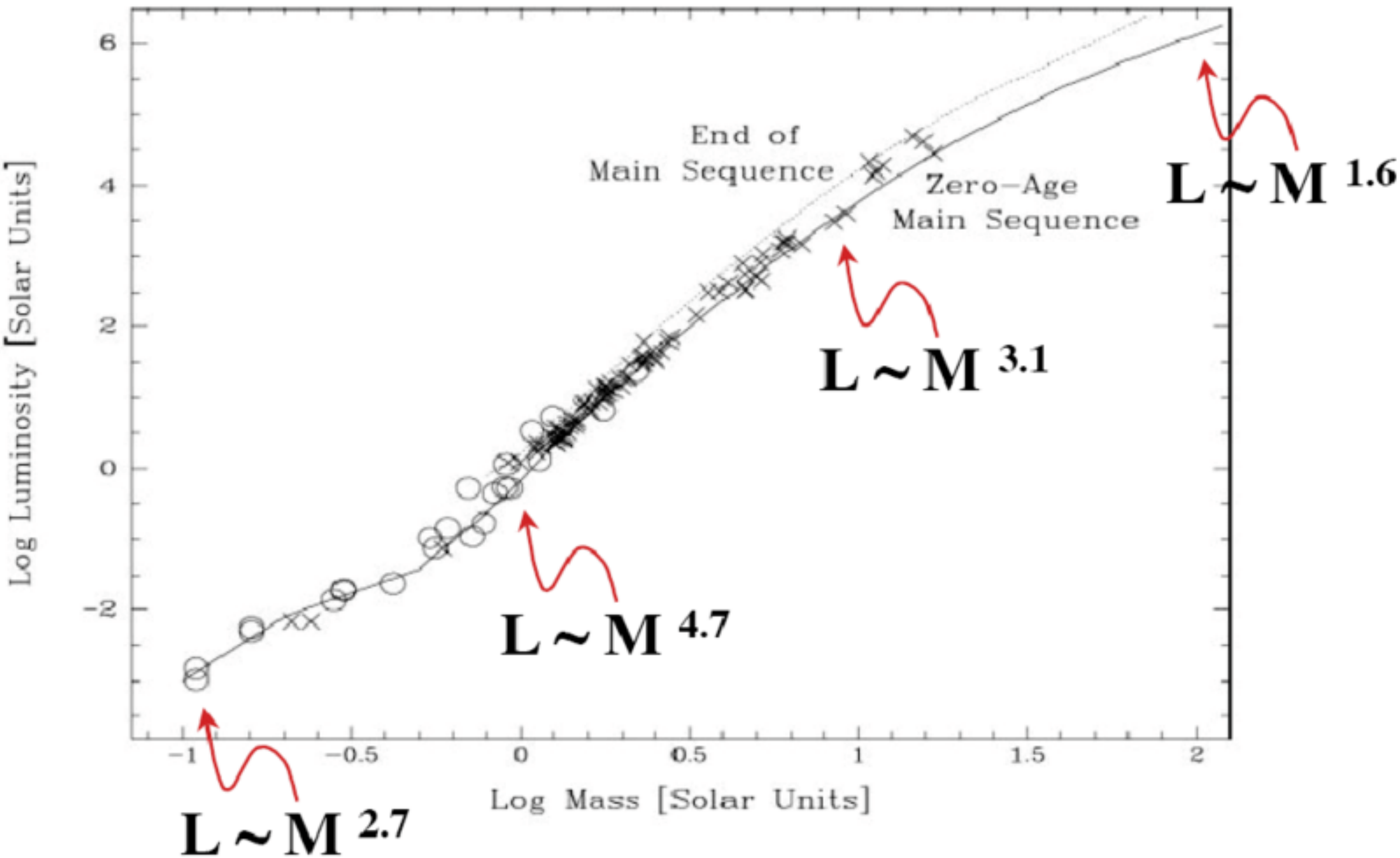
I. Mass-Luminosity relationship

$$\frac{L}{4\pi r^2} = -\frac{c}{3\kappa\rho}\nabla aT^4$$

Use dimensional arguments; keep only the non-constant terms,

$$\frac{L}{4\pi r^2} = -\frac{c}{3\kappa\rho}\nabla aT^4 \rightarrow \frac{L}{R^2} \propto \frac{R^3 M^4}{\kappa M R^5} \rightarrow L \propto \frac{M^3}{\kappa}$$

Scale to the Sun $$\left(\frac{L}{L_s}\right) = \left(\frac{\kappa_s}{\kappa}\right)\left(\frac{M}{M_s}\right)^3$$



End of
Main Sequence
Zero-Age
Main Sequence
Log Luminosity [Solar Units]
Log Mass [Solar Units]
6
4
2
0
-2
-1
-0.5
0
0.5
1
1.5
2
L ~ M 1.6
L ~ M 3.1
L ~ M 4.7
L ~ M 2.7

IIa. Main Sequence Upper Mass Limit

$$\frac{L}{4\pi r^2} = -\frac{c}{3\kappa\rho}\nabla aT^4 = -\frac{c}{3\kappa\rho}\nabla(3P) \rightarrow \nabla P = -\frac{\kappa\rho L}{4\pi c r^2}$$

$$\nabla P = -\frac{GM}{r^2}\rho \rightarrow \nabla P = -\frac{\kappa\rho L}{4\pi c r^2} > -\frac{GM\rho}{r^2}$$

$$\rightarrow L > \frac{4\pi c G M}{\kappa} \rightarrow$$ Unstable, Eddington Limit

$$L = \left(\frac{M}{M_s}\right)^{3.1} L_s = \frac{4\pi c G M}{\kappa} \rightarrow \left(\frac{M}{M_s}\right)^{2.1} = \frac{4\pi c G}{\kappa}\left(\frac{M_s}{L_s}\right)$$

$$\rightarrow \left(\frac{M}{M_s}\right)^{2.1} = 38{,}000 \rightarrow \left(\frac{M}{M_s}\right) \approx 150$$

A little high, but in the right ballpark. If not the direct reason likely the correct reason is related.

IIb. Main Sequence Lower Mass Limit

$$\rho \frac{d^2 r}{dt^2} = -\nabla P(r) - G\frac{M(r)}{r^2}\rho(r)$$

A protostar has not yet ignited nuclear fusion so, as it radiates, it cools and contracts compressing and heating. Eventually, it compresses and heats to the point where fusion ignites in its core and it becomes a star. This happens when its core temperature reaches around 10 million K.

As a protostar contracts and heats, a complication may set in. If the protostar becomes compact enough, quantum mechanical effects become important and we must take account of the Pauli exclusion principle, degenerate gas pressure becomes important. This happens when uncertainty principle effects,

$$\Delta x \Delta p \approx \hbar$$

are non-negligible. When does this happen?

For a gas with number density *n* and thermal energy *kT*, estimate when this happens.

- *n* implies average particle separation of $n^{-1/3}$
- *kT* implies average momentum of $p = (3m_e kT)^{1/2}$

Quantum effects will be important when

$$\Delta x \Delta p = \Delta x \sqrt{3m_e kT} = \hbar \rightarrow \Delta x = \frac{\hbar}{\sqrt{3m_e kT}} \approx n_e^{-1/3}$$

Quantum effects will be important when

$$\Delta x = \frac{\hbar}{\sqrt{3m_e kT}} \approx 5.2\times10^{-10}\left(\frac{m}{m_e}\right)^{-1/2}\left(\frac{T}{10^7 K}\right)^{-1/2} cm$$

$$\rightarrow n_e \approx 1.8x10^{27}\left(\frac{m}{m_e}\right)^{3/2}\left(\frac{T}{10^7 K}\right)^{3/2} cm^{-3}$$

$$\rightarrow \rho \approx 3{,}000\left(\frac{m}{m_e}\right)^{3/2}\left(\frac{T}{10^7 K}\right)^{3/2} g\ cm^{-3}$$

The central ***ρ*** of a ***0.1 M$_\odot$*** star is ~ ***500 g cm^{-3}***. (i) For smaller ***M***, ***ρ*** is larger and, (ii) Brown dwarfs, failed stars, have ***T ~ 10^6 K***. Detailed modeling finds that QM effects are important in stars below ***0.08 M$_\odot$***.

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2. We make dimensional arguments to see if we can understand some properties of Main Sequence stars.

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III. $$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$$

1(i). What does an equilibrium, spherically symmetric, constant density star look like?

I. Mass Distribution

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho$$

$$\rightarrow \int^r dM(r) = M(r) = 4\pi \int^r r^2 \rho_o dr = \frac{4\pi}{3}\rho_o r^3$$

$$\rightarrow \boxed{M_* = \frac{4\pi}{3}\rho_o R_*^3}$$

II. Momentum Conservation

$$\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho_o = -\frac{4\pi r G\rho_o^2}{3}$$

$$\rightarrow \int^r dP(r) = P(r) - P_c = -\frac{4\pi G\rho_o^2}{3}\int^r r\,dr = -\frac{2\pi}{3}G\rho_o^2 r^2$$

$$\rightarrow P(r) = P_c - \frac{2\pi}{3}G\rho_o^2 r^2$$

Estimate the central pressure. Let r go to R_*,

$$\rightarrow P_c = \frac{2\pi}{3}G\rho_o^2 R_*^2 = \left(\frac{3}{8\pi}\right)\left(\frac{GM_*^2}{R_*^4}\right)$$

Central Pressure and Temperature

$$P_c = G\left(\frac{3}{8\pi}\right)\left(\frac{M_*^2}{R_*^4}\right)$$

Estimate T_c. Use the perfect gas law, $P = (\rho/\mu m_o)kT$, and Solar parameters to find,

$$P_c = G\left(\frac{3}{8\pi}\right)\left(\frac{M_*^2}{R_*^4}\right) = \left(\frac{\rho_o}{\mu m_o}\right)kT_c$$

$$\rightarrow \boxed{T_c = \left(\frac{\mu m_o}{2k}\right)\left(\frac{GM_*}{R_*}\right) = 7.1\times10^6\,K}$$

III. Energy Conservation

$$\rightarrow \frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$$

We can't make further progress unless we know the nuclear energy generation process.

1(ii). What does a star with spherically symmetric, density distribution, $\rho_o(1-r/R_*)$, look like?

I. Mass Distribution

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho_o (1 - \frac{r}{R_*})$$

$$\rightarrow M(r) = 4\pi \int^r r^2 \rho_o (1 - \frac{r}{R_*}) dr = \frac{4\pi}{3} \rho_o r^3 (1 - \frac{3r}{4R_*})$$

$$\rightarrow M_* = \frac{\pi \rho_o R_*^3}{3}$$

II. Momentum Conservation

$$\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho_o(1-\frac{r}{R_*})$$

$$\rightarrow P(r) - P_c = -\frac{4\pi G\rho_o^2}{3}\int^r r(1-\frac{3r}{4R_*})(1-\frac{r}{R_*})dr$$

$$\rightarrow P = P_c - \frac{2\pi G\rho_o^2 r^2}{3}(1-\frac{7r}{6R_*}+\frac{3r^2}{8R_*^2})$$

$$\rightarrow P_c = \frac{5\pi}{36}G\rho_o^2 R_*^2 = \frac{5}{4\pi}\frac{GM_*^2}{R_*^4}$$

Central Pressure and Temperature

$$P_c = G\left(\frac{5}{4\pi}\right)\left(\frac{M_*^2}{R_*^4}\right)$$

Estimate T_c. Use the perfect gas law, *$P = (\rho/\mu m_o)kT$*, and Solar parameters to find,

$$P_c = G\left(\frac{5}{4\pi}\right)\left(\frac{M_*^2}{R_*^4}\right) = \left(\frac{\rho_o}{\mu m_o}\right)kT_c \rightarrow T_c = \left(\frac{5\mu m_o}{3k}\right)\left(\frac{GM_*}{R_*}\right)$$

$$\rightarrow T_c = 23\times10^6\,K$$