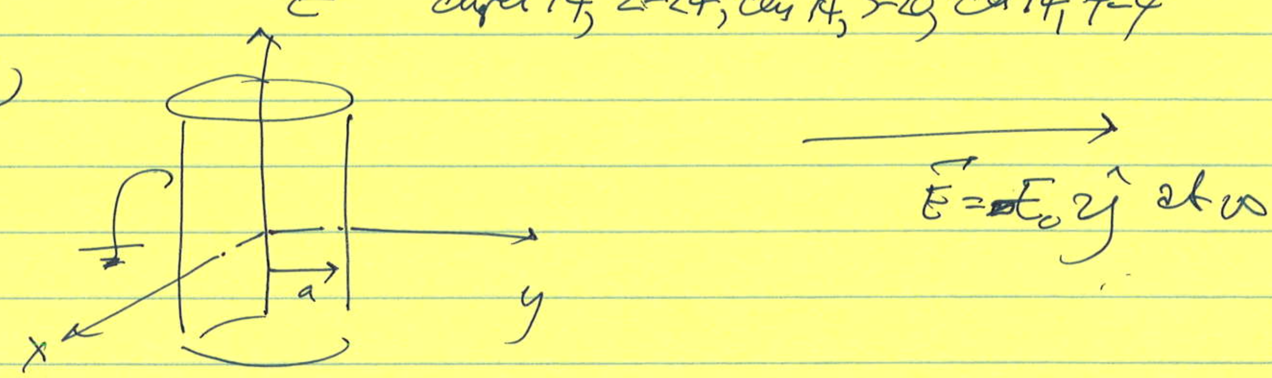
Homework 2; Map, Chapter 14, 11-5; Chapter 14, 11-1; Ch 14, 11-35, Chapter 14, 2-24; Ch 14, 3-29; Ch 14, 4-4

(a)



$a = 1$

(i) $f(z) = z + \frac{1}{z} = u(x,y) + i v(x,y)$

a) find $u(x,y)$, $v(x,y)$

$$f(z) = (x+iy) + \frac{1}{(x+iy)} = (x+iy) + \frac{x-iy}{(x^2+y^2)}$$

$$\Rightarrow \boxed{u(x,y) = x + \frac{x}{x^2+y^2}}, \quad \boxed{v(x,y) = y - \frac{y}{x^2+y^2}}$$

(ii) for our chosen BCs $\Rightarrow \Phi = 0$ at $a = 1$ and

$$\Phi = + E_0 y \text{ at } \infty$$

from BC at ∞, we see that $v(x,y)$ might work,

$$v(x,y) \rightarrow y \text{ as } (x^2+y^2) \rightarrow \infty$$

(iii) at $x^2 + y^2 = a^2 = 1 \Rightarrow v(x,y) = y\left(\frac{x^2+y^2-1}{x^2+y^2}\right)$

$= 0$!

$\Rightarrow$ our map $f(z) = z + \frac{1}{z}$ is the appropriate choice for the problem stated.

(iv) $\Phi(x,y) = \left(y - \frac{y}{x^2+y^2}\right) E_0$

and

$$\vec{E}(x,y) = -\hat{y} E_0 + \frac{E_0 \hat{y}}{x^2+y^2} - \frac{2y^2 E_0 \hat{y}}{(x^2+y^2)^2}$$

$$= -E_0 \hat{y} \left\{ 1 - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} \right\}$$

(10) Chapter 14, 11-5

find the residues of

$$\frac{z^{1/3}}{(1+z^2)} = \frac{z^{1/3}}{(1+iz)(1-iz)} = \frac{z^{1/3}}{i^2(z-i)(z+i)} = +\frac{z^{1/3}}{(z-i)(z+i)}$$

$\Rightarrow$ poles at $z = i, -i$

a) $R(i) = (z-i)\left[+\frac{z^{1/3}}{(z-i)(z+i)}\right], \; z \to i$

$$= +\frac{(i)^{1/3}}{2i} = +\frac{1}{2i}(\cos\theta + i\sin\theta)^{1/3}, \; \theta = \frac{\pi}{2}$$

$$= \frac{-i}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$\boxed{R(i) = \frac{1}{2}\sin\frac{\pi}{6} - \frac{i}{2}\cos\frac{\pi}{6} = \frac{1}{4} - i\frac{\sqrt{3}}{4}}$$

b) $R(-i) = (z+i)\left[\frac{+z^{1/3}}{(z-i)(z+i)}\right]$ as $z \to -i$

$$= +\frac{(-i)^{1/3}}{(z-i)} \to +\frac{(-i)^{1/3}}{(-2i)} = +\frac{i}{2}(-i)^{1/3}$$

$$= +\frac{i}{2}(\cos\theta + i\sin\theta)^{1/3}(\cos\theta_1 + i\sin\theta_1)^{1/3}$$

$\Rightarrow \theta = +\frac{\pi}{2}, \; \theta_1 = \pi$

$$= \frac{i}{2}(i)^{1/3}$$

$$= \frac{i}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= \frac{i}{2}\left(\cos\frac{2\pi}{6} - \sin\frac{2\pi}{6} + 2i\sin\frac{\pi}{6}\right.$$

$$\frac{1}{3} = +\frac{i}{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= +\frac{i}{2}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$

$$= +\frac{1}{2}\sin\frac{\pi}{6} - \frac{i}{2}\cos\frac{\pi}{6}$$

$$R(-i) = \frac{i}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\frac{\sqrt{3}}{2} \qquad \frac{1}{2} \qquad \frac{1}{2} \qquad \frac{\sqrt{3}}{2}$$

$$= \frac{i}{2}\left[\left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) + i\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} + i\frac{1}{2}\frac{1}{2}\right]$$

$$= \frac{i}{2}\left[i\frac{3}{4} + i\frac{1}{4}\right]$$

$$= \frac{i}{2}(i)$$

$$R(-i) = -\frac{1}{2}$$

(2) Chapter 14, 11-11

for $f(z) = \frac{e^z}{1-z}$, find Laurent series for $|z|<1$ and $|z|>1$ w/center $z_0 = 0$ for "circles."

Soln

(i) $|z|<1$:

$$f(z) = \frac{e^z}{1-z} = \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(1 + z + z^2 + z^3 + \cdots\right)$$

$$= 1 + z\left(1 + \frac{1}{1!}\right) + z^2\left(1 + \frac{1}{1!} + \frac{1}{2!}\right) + z^3\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}\right) + \cdots$$

$$= \sum_{n=0}^{\infty} z^n \sum_{p=0}^{n} \frac{1}{p!}$$

Let $a_n = \sum_{p=0}^{n} \frac{1}{p!}$

which is convergent when $|z|<1$

(ii) $|z|>1$

$$f(z) = \frac{e^z}{1-z} = \frac{e^z}{-z\left(1-\frac{1}{z}\right)}$$

$$= \left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)\left(-\frac{1}{z}\right)\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

$$= -\frac{1}{z}\Big[\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)+\left(z+1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$+\left(\frac{z^2}{2!}+\frac{z}{2!}+\frac{1}{2!}+\frac{1}{2!z}+\cdots\right)$$

$$+\left(\frac{z^3}{3!}+\frac{z^2}{3!}+\frac{z}{3!}+\frac{1}{3!}+\frac{1}{3!z}+\cdots\right)$$

$$+\cdots\Big]$$

$$= -\frac{1}{z}\Big[\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\frac{1}{n!}+\cdots\right)$$

$$+\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)\frac{1}{z}+\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)\frac{1}{z^2}$$

$$+\cdots\sum_{p=0}^{N}\frac{1}{p!}\frac{1}{z^n}+\cdots\Big]$$

$$-\frac{1}{z}\Big[\left(1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots\right)z+\left(\frac{1}{2!}+\frac{1}{3!}+\cdots\right)z^2$$

$$+\left(\frac{1}{3!}+\frac{1}{4!}+\cdots\right)z^3+\cdots+\left(\frac{1}{n!}+\frac{1}{(n+1)!}+\cdots\right)z^n$$

$$+\cdots+\sum_{p=n}^{N}\frac{1}{p!}z^n+\cdots\Big]$$

$$f(z) = \sum_{k=0}^{N} \left(\sum_{p=0}^{N} \frac{1}{p!} \right) \frac{-1}{z^{k+1}} - \sum_{k=0}^{N} \left(\sum_{p=k+1}^{N} \frac{1}{p!} \right) z^k$$

<u>note:</u> $e^1 = 1 + 1 + \frac{1}{2!} 1^2 + \frac{1}{3!} 1^3 + \cdots$

$$= \sum_{p=0}^{\infty} \frac{1}{p!}$$

$$\Rightarrow f(z) = \sum_{k=0}^{\infty} \frac{-e}{z^{k+1}} - \sum_{k=0}^{\infty} \left(\sum_{p=k+1}^{\infty} \frac{1}{p!} \right) z^k$$

<u>note:</u> (a) $\sum_{p=k+1}^{\infty} \left(\frac{1}{p!} \right) = \left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \right), k=0$

$+ \left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \right), k=1$

$+ \left(\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots \right), k=2$

(b) Let's change the summation limit to something more sensible.

$\sum_{k=0}^{N} \left(\sum_{p=k+1}^{N} \frac{1}{p!} \right) z^k$ is our summation.

Write $\sum_{k=0}^{N} \left\{ \sum_{p=1}^{N} \left(\frac{1}{p!} \right) z^k - \sum_{p=0}^{k} \frac{1}{p!} z^k \right\}$

and we have,

$$f(z) = \sum_{k=0}^{\infty} \frac{-e}{z^{k+1}} - \sum_{k=0}^{\infty} \left[\sum_{p=1}^{\infty} \frac{1}{p!} z^k - \sum_{p=0}^{k} \frac{1}{p!} z^k \right]$$

(c) the first series $\to 0$ at infinity and is convergent, is the second series convergent

$$-\sum_{k=0}^{\infty}\left\{\left(e-\sum_{p=0}^{k}\frac{1}{p!}\right)z^{k}\right\}$$

<u>ratio test:</u>

$$\left|\frac{\left(e-\sum_{p=0}^{n+1}\frac{1}{p!}\right)z^{n+1}}{\left(e-\sum_{p=0}^{n}\frac{1}{p!}\right)z^{n}}\right|$$

$$\left|\frac{e-\sum^{n+1}\frac{1}{p!}}{e-\sum^{n}\frac{1}{p!}}\right|\left|z\right|$$

as $n\to\infty$ $\sum^{n+1}\frac{1}{p!}\to e$ and $\left|\frac{e-\sum^{n+1}\frac{1}{p!}}{e-\sum^{n}\frac{1}{p!}}\right|\downarrow$

as $n\uparrow$

$\Rightarrow$ as long as $|z|<\infty$, we will be okay.