Homework #3

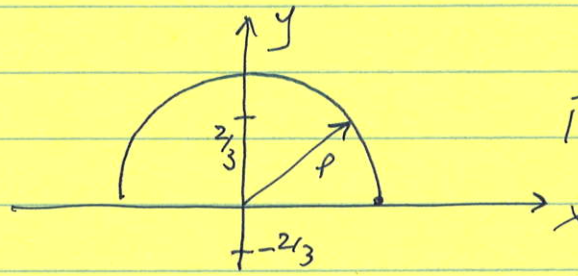
16) <u>find</u>: $\int_0^{\infty} \frac{\cos 2x}{9x^2+4}\,dx$

<u>Solve</u> $\oint \frac{\cos 2z}{9z^2+4}\,dz = \oint \frac{e^{i2z}+e^{-i2z}}{2(3z-2i)(3z+2i)}\,dz$

$= \frac{1}{18}\oint \frac{e^{i2z}-e^{-i2z}}{\underbrace{(z-\frac{2i}{3})(z+\frac{2i}{3})}}\,dz$

poles at $z = \frac{2i}{3}, -\frac{2i}{3}$

a) <u>Solve 2 contour integrals</u>



$\frac{1}{18}\oint \frac{e^{i2z}\,dz}{(z-\frac{2i}{3})(z+\frac{2i}{3})}$

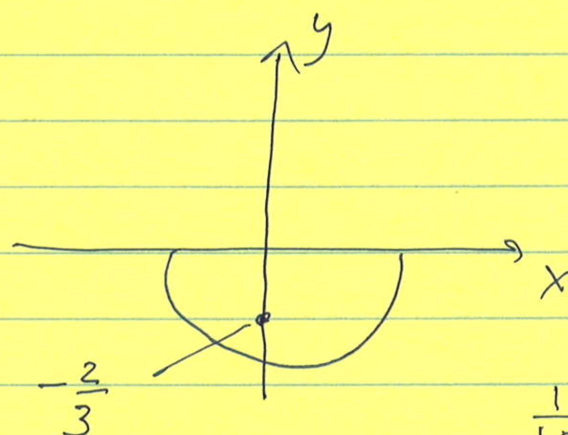
1 pole at $z = \frac{2}{3}i$

$\Rightarrow \frac{1}{18}\oint = \frac{1}{18} \times 2\pi i \lim_{z\to\frac{2i}{3}} \left(\frac{e^{2iz}}{z+\frac{2i}{3}}\right)$

$= \frac{\pi i}{9}\left[\frac{e^{-4/3}}{\frac{4i}{3}}\right]$

$= \frac{\pi}{12} e^{-4/3}$

b)



$\frac{1}{18}\oint \frac{e^{-i2z}\,dz}{(z-\frac{2i}{3})(z+\frac{2i}{3})}$

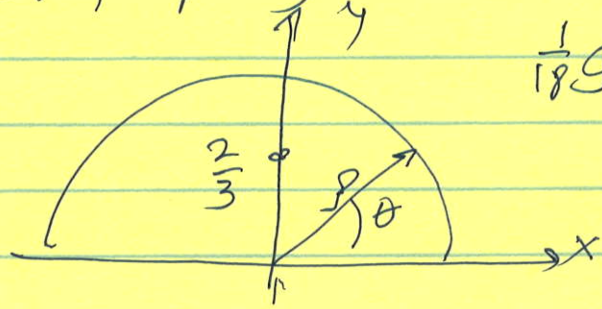
1 pole at $z = -\frac{2i}{3}$

$\frac{1}{18}\oint = \frac{1}{18} 2\pi i \lim_{z\to-\frac{2i}{3}} \left(\frac{e^{-i2z}}{z-\frac{2i}{3}}\right)$

$= \frac{\pi}{9} i \left[\frac{e^{-4/3}}{-4i/3}\right]$

$= -\frac{\pi}{12} e^{-4/3}$

a) Next, explicitly solve



$$\frac{1}{18}\oint \frac{e^{i2z}\,dz}{(z-\frac{2i}{3})(z+\frac{2i}{3})} \leftarrow i\rho e^{i\theta}d\theta$$

$$= \frac{1}{18}\int_0^{\pi} \frac{e^{2i\rho e^{i\theta}}\, i\rho e^{i\theta}\,d\theta}{(\rho e^{i\theta}-\frac{2i}{3})(\rho e^{i\theta}+\frac{2i}{3})}$$

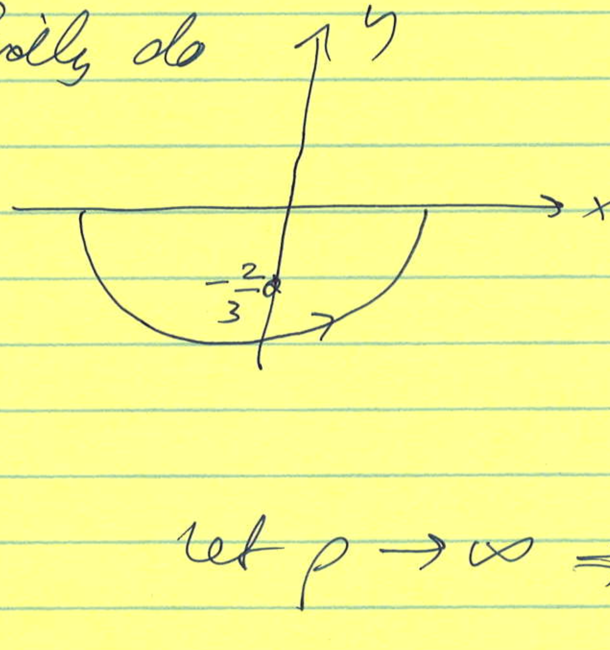
$$+\frac{1}{18}\int_{-\rho}^{\rho} \frac{e^{i2x}\,dx}{(x-\frac{2i}{3})(x+\frac{2i}{3})}$$

$$= \frac{i\rho}{18}\int_0^{\pi} \frac{e^{2i\rho\cos\theta}\, e^{i\theta}\, e^{-2\rho\sin\theta}\,d\theta}{(\rho e^{i\theta}-\frac{2i}{3})(\rho e^{i\theta}+\frac{2i}{3})}$$

$$+\frac{1}{18}\int_{-\rho}^{\rho} \frac{e^{2ix}\,dx}{(x^2+\frac{4}{9})}$$

let $\rho \to \infty$ $\frac{1}{18}\oint \to \frac{1}{18}\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+\frac{4}{9}}dx$

b) finally do



$$\frac{1}{18}\oint \frac{e^{-i2z}\,dz}{(z-\frac{2i}{3})(z+\frac{2i}{3})}$$

$$= \frac{1}{18}\int_{\pi}^{2\pi} \frac{e^{-2iz}\,dz}{(z-\frac{2i}{3})(z+\frac{2i}{3})}$$

$$+\frac{1}{18}\int_{+\rho}^{-\rho} \frac{e^{-2ix}}{(x^2+\frac{4}{9})}dx$$

let $\rho \to \infty \Rightarrow \frac{1}{18}\oint = \frac{1}{18}\int_{+\infty}^{-\infty} \frac{e^{-2ix}}{(x^2+\frac{4}{9})}dx$

c) Continue $\Rightarrow$ ~~$\frac{1}{18}\oint$~~

Because we want cos 2X,
change direction of contour

$$\frac{1}{18}\int_{-\infty}^{\infty}\frac{e^{2ix}dx}{\left(x^2+\frac{4}{9}\right)} + \frac{1}{18}\int_{-\infty}^{\infty}\frac{e^{-2ix}}{\left(x^2+\frac{4}{9}\right)}dx$$

$$= \frac{1}{18}\int_{-\infty}^{\infty}\frac{e^{2ix}+e^{-2ix}}{\left(x^2+4/9\right)}dx$$

$$= \frac{1}{9}\int_{-\infty}^{\infty}\frac{\cos 2x}{x^2+\frac{4}{9}}dx$$

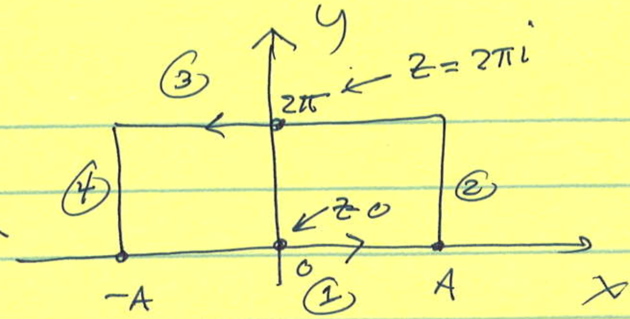
$$= \frac{2}{9}\int_{0}^{\infty}\frac{\cos 2x}{x^2+\frac{4}{9}}dx$$

$$= \frac{\pi}{12}e^{-\frac{4}{3}} - \left(-\frac{\pi}{12}e^{-\frac{4}{3}}\right)$$

$$= \frac{\pi}{6}e^{-4/3}$$

$$\Longrightarrow \boxed{\int_{0}^{\infty}\frac{\cos 2x\,dx}{x^2+\frac{4}{9}} = \frac{3\pi}{4}e^{-4/3}}$$

(17) (A) find $\oint \frac{e^{pz}}{1-e^{z}}\, dz$ around the contour



(i) there is a pole at $\underline{z=0} \Rightarrow 1-e^{z} \to 0$

(ii) there is a second pole at $\underline{z = 2\pi i} \Rightarrow 1-e^{z} = 1-e^{2\pi i}$
$= 1-1 = 0$

(iii) find the residues,

a) $z\frac{e^{pz}}{1-e^{z}}$ as $z \to 0$ $= \frac{ze^{pz}}{1-e^{-z}}$; indeterminate

$$\Rightarrow \frac{e^{pz}+pze^{pz}}{-e^{z}} \to -1$$

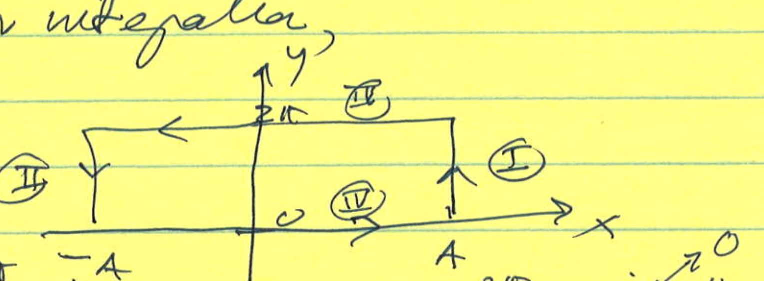
b) $(z-2\pi i)\frac{e^{pz}}{1-e^{z}}$ as $z \to 2\pi i$

$$= \frac{e^{pz}+p(z-2\pi i)e^{pz}}{-e^{z}} \to \frac{e^{2\pi i p}}{-1} = -e^{2\pi i p}$$

$$\boxed{\oint = 2\pi i\left[-\frac{1}{2} - \frac{e^{2\pi i p}}{2}\right] = -i\pi\left(1+e^{2\pi i p}\right)}$$

(B) Perform the contour integration,

$$\oint \frac{e^{pz}\,dz}{1-e^{z}}$$



$$\text{(I)}\quad \int_0^{2\pi} \frac{e^{A}e^{ipy}\, i\,dy}{1-e^{iy}e^{A}} = \int_0^{2\pi} \frac{e^{ipy}\, i\,dy}{e^{-A}-e^{iy}} \xrightarrow[A\to\infty]{} -\int_0^{2\pi} i\,dy\, e^{i(p-1)y} \to 0$$

$$\text{(II)}\quad \int_{2\pi}^{0} \frac{e^{-A}e^{ipy}\, i\,dy}{1-e^{iy}e^{-A}} = \int_{2\pi}^{0} \frac{e^{ipy}\, i\,dy}{e^{A}-e^{iy}} \xrightarrow[A\to\infty]{} 0$$

$$\text{(III)}\quad \int_{A}^{-A} \frac{e^{ip2\pi}e^{px}\,dx}{1-e^{i2\pi}e^{x}} = e^{i2\pi p}\int_{A}^{-A} \frac{e^{px}\,dx}{1-e^{x}} \to e^{2\pi i p}\int_{\infty}^{-\infty} \frac{e^{px}}{1-e^{x}}\,dx$$

$$\text{(IV)}\quad \int_{-A}^{A} \frac{e^{px}\,dx}{1-e^{x}} \to \int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx$$

$$\text{I} + \text{II} + \text{III} + \text{IV} = \int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx\left[1-e^{2\pi i p}\right] = -i\pi\left[1+e^{2\pi i p}\right]$$

$$\int_{-\infty}^{\infty} \frac{e^{px}\,dx}{1-e^{x}} = -i\pi\left[\frac{1+e^{2\pi i p}}{1-e^{2\pi i p}}\right]$$

$$= -i\pi\left[\frac{e^{-\pi i p}+e^{\pi i p}}{e^{-\pi i p}-e^{\pi i p}}\right]$$

$$\boxed{\int_{-\infty}^{\infty} \frac{e^{px}\,dx}{1-e^{x}} = i\pi\left(\frac{\cos \pi p}{\sin \pi p}\right)}$$

(18)

$$\boxed{\begin{aligned} &(7.18)\quad PV\int_{-\infty}^{\infty} \frac{u(x)}{x-a}\,dx = -\pi v(a) \\ &PV\int_{-\infty}^{\infty} \frac{v(x)}{x-a}\,dx = +\pi u(a) \end{aligned}}$$

(a) given $f(x) = u(x) + i v(x)$

even function of x ; odd function of x

take $f^*(x) = u(x) - i v(x)$; u, v, x are real

$$f^*(-x) = u(-x) + i v(-x)$$
$$= u(x) \cancel{+} i v(x)$$

$u(x)$ is even in x ; $v(x)$ is odd in x

$$\Rightarrow \boxed{f^*(x) = f(-x)}$$

(b) show that

$$PV\int_0^{\infty} \frac{2x\, v(x)}{(x^2-a^2)}\,dx = \pi u(a)$$

and

$$PV\int_0^{\infty} \frac{2a\, u(x)}{(x^2-a^2)}\,dx = \pi v(a)$$

(A) Solve $\oint \frac{2z f(z)\,dz}{(z^2-a^2)}$ by contour integration

$$= \oint 2z f(z)\left[\frac{+\frac{1}{2a}}{(z-a)} + \frac{-\frac{1}{2a}}{(z+a)}\right]dz$$

$$= \frac{1}{a}\oint\left\{\frac{+zf(z)}{(z-a)} - \frac{zf(z)}{(z+a)}\right\}dz$$

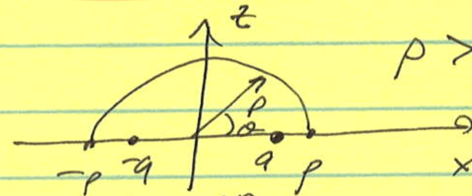
simple pole at $z=a$ simple pole at $z=-a$

Contour Integral yields (and semicircle w/ radius $\rho \to \infty$)

$$\oint \frac{2zf(z)\,dz}{(z^2-a^2)} = \frac{2\pi i}{a}\left[+\left(\frac{a f(a)}{2}\right) - \left(\frac{-a f(-a)}{2}\right)\right]$$

$$\boxed{= \pi i\left[f(a)+f(-a)\right] = 2\pi i\, u(a)}$$

(B) $\oint \frac{2zf(z)\,dz}{(z^2-a^2)}$ over



$\rho > a$

$$= \frac{1}{a}\int_0^\pi \frac{\rho e^{i\theta} f(z)}{\rho e^{i\theta}-a}\, i\rho e^{i\theta}\,d\theta - \frac{1}{a}\int \frac{\rho e^{i\theta} f(z)}{\rho e^{i\theta}+a}\, i\rho e^{i\theta}\,d\theta$$

semi-circle

$$+\frac{1}{a}\int_{-\rho}^{\rho}\frac{x f(x)}{(x-a)}\,dx - \frac{1}{a}\int_{-\rho}^{\rho}\frac{x f(x)}{(x+a)}\,dx$$

if $f(z) \to 0$ as $z \to \infty$ fast enough, the $\cap$ line integrals $\to 0$ for $\rho \to \infty$

and so, ← adding the 2 integrals

$$\Rightarrow \oint \frac{2z f(z)\,dz}{(z^2-a^2)} = \int_{-\infty}^{\infty} \frac{2x f(x)}{(x^2-a^2)}dx = \pi i u(a)$$

↑ from the contour integration

recall: $f(z) = u(x,y) + i v(x,y)$

for $y=0$, $f(z) = u(x,0) + i v(x,0)$

and we have that

$$\int_{-\infty}^{\infty} \frac{2x i v(x)\,dx}{(x^2-a^2)} = 2\pi i u(a)$$

$$\boxed{\int_{-\infty}^{\infty} \frac{2x v(x)\,dx}{(x^2-a^2)} = 2\pi u(a)}$$

$$\int_{0}^{\infty} \frac{2x v(x)\,dx}{(x^2-a^2)} = \pi u(a)$$