HW #7

3-9-8, 3-9-14, 3-9-15, 3-15-25, 3-15-27, 3-15-34;
show that $Tr\, A^{-1}MA = Tr\, M$
and
$$|A^{-1}MA| = |M|$$

f2 Prob 3-9-8

a) Verify that $(AB)^{+} = B^{+}A^{+}$

$$\boxed{(AB)^{+}_{ik}} = (A^{*}B^{*})_{ki} \qquad \text{"}+\text{"} = \text{conjugate transpose}$$

$$= (A^{*}_{kj}\, B^{*}_{ji})$$

$$= A^{*T}_{jk}\, B^{*T}_{ij} = B^{*T}_{ij}\, A^{*T}_{jk} \boxed{= (B^{+}A^{+})_{ik}}$$

b) Verify (9.11) $(ABCD)^{T} = D^{T}C^{T}B^{T}A^{T}$

$$\boxed{(ABCD)^{T}} = D^{T}(ABC)^{T} = D^{T}C^{T}(AB)^{T}$$

$$\boxed{= D^{T}C^{T}B^{T}A^{T}}$$

43. 3-9-14

Use (9.11) and (9.12) to simplify $(AB^TC)^T$; $(C^{-1}MC)^{-1}$; $(AH)^{-1}(AHA^{-1})^3(HA^{-1})^{-1}$

(i) $(AB^TC)^T = C^T(AB^T)^T = C^TBA^T$

(ii) $(C^{-1}MC)^{-1} = C^{-1}M^{-1}C$

(iii) $(AH)^{-1}(AHA^{-1})^3(HA^{-1})^{-1}$

$= (H^{-1}A^{-1})(AHA^{-1})(AHA^{-1})(AHA^{-1})(AH^{-1})$

$= H^{-1}(A^{-1}A)H(A^{-1}A)H(A^{-1}A)H(A^{-1}A)H^{-1}$

$= (H^{-1}\cancel{A})HH(AH^{-1})$

$= (H^{-1}H)H\overleftrightarrow{\mathbb{1}}$

$= \overleftrightarrow{\mathbb{1}}H\overleftrightarrow{\mathbb{1}}$

44. 3-9-15, Pauli Spin Matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) Show A, B, C are Hermitian. $(H = H^{\dagger})$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^{\dagger} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (A^*)^T$$

$$B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad B^{\dagger} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = (B^*)^T$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (C^*)^T$$

b) Show that

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

(i) $$[B,C] = \underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}_{BC} - \underbrace{\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}}_{CB} = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix}$$

$$[A,[B,C]] = \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix} - \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix} = \begin{pmatrix} \cancel{2i} & 0 \\ 0 & \cancel{2i} \end{pmatrix}$$

(ii) $$[C,A] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$[B,[C,A]] = \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix} - \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(iii) $$[A,B] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

$$[C,[A,B]] = \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix} - \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

c) $\underbrace{[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0}_{\text{show}}$

(i) $[A,[B,C]] = [A, BC - CB]$

$= (ABC - ACB) - (BCA - CBA)$

(ii) $[B,[C,A]] = [B, CA - AC]$

$= (BCA - BAC) - (CAB - ACB)$

(iii) $[C,[A,B]] = [C, AB - BA]$

$= (CAB - CBA) - (ABC - BAC)$

Combing $\Rightarrow 0$, as long as matrices are conformable

(45) 3-15-25

$$M = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$$

find C that diagonalizes M, that is, find $C \ni$

$$C^{-1}MC = D$$

Sol^{n}

(a) find eigenvalues of M

$$Mx = \lambda x \rightarrow \begin{pmatrix} 1-\lambda & 0 \\ 3 & -2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

need $\begin{vmatrix} (1-\lambda) & 0 \\ 3 & -(2+\lambda) \end{vmatrix} = -(1-\lambda)(2+\lambda) = 0$

$$\Rightarrow \boxed{\lambda = 1, -2}$$

(b) Substitute λ into characteristic value equation to find:

$\lambda = 1$: $x_1 + 0y_1 = 1x_1$ $\quad$ $\lambda = -2$: $x_2 + 0y_2 = -2x_2$

$3x_1 - 2y_1 = 1y_1$ $\quad$ $3x_2 - 2y_2 = -2y_2$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

ad

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

(c) find eigenvectors (x_1, y_1), (x_2, y_2)

$\underline{\lambda = 1}$: $\rightarrow x_1 = y_1$ $\qquad$ $\underline{\lambda = -2}$: $x_2 = 0$, $y_2 =$ anything

$\Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $\Rightarrow \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 \end{pmatrix}}$$

C that diagonalizes M through a similarity transformation

(46) 3-15-27

Express $x^2 + y^2 - 5z^2 + 4xy = 15$ in a coordinate system defined as the principal axis system.

Sol^n

a) Find the coordinate system by finding the eigenvectors of matrix defined by the above equation. Rewrite the above as

$$(x \; y \; z)\underbrace{\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}}_{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 15$$

$\Rightarrow m_{11} = 1, \; m_{22} = 1, \; m_{33} = -5$

assume M is symmetric.

$\Rightarrow m_{12} = m_{21} = 2, \; m_{13} = m_{31} = 0, \; m_{23} = m_{32} = 0$

and

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

b) Find eigenvalues of M

$$\begin{vmatrix} (1-\lambda) & 2 & 0 \\ 2 & (1-\lambda) & 0 \\ 0 & 0 & -(5+\lambda) \end{vmatrix} = -(1-\lambda)^2(5+\lambda) + 4(5+\lambda) = 0$$

$$(5+\lambda)\left[4 - (1-\lambda)^2\right] = 0$$

$$\Rightarrow \lambda = -5 \text{ and } 4-(1-\lambda)^2 = 0 = 4 - \left[1 - 2\lambda + \lambda^2\right]$$

$$= -\lambda^2 + 2\lambda + 3$$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda-3)(\lambda+1)$$

$$\Rightarrow \lambda = -1, 3$$

and $\boxed{\lambda = 3, -1, -5}$

b) $$\Rightarrow \begin{pmatrix} +1 & 2 & 0 \\ 2 & +1 & 0 \\ 0 & 0 & -5 \end{pmatrix} \overbrace{\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}}^{\psi} = \overbrace{\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}}^{\psi} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\boxed{\begin{gathered} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -5 \end{pmatrix} = \psi \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{pmatrix} \psi^{-1} \\ \text{and} \\ \psi^{-1} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -5 \end{pmatrix} \psi = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{pmatrix} \end{gathered}}$$

diagonalizes matrix (rotates coordinates to eigenvector directions)

$$\Rightarrow (x' \; y' \; z') \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = 15$$

length invariant after coordinate rotation

$$\boxed{3x'^2 - y'^2 - 5z'^2 = 15}$$

47 3-15-34

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) $$\begin{cases} A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ C^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

b) $$\begin{cases} A^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A \\ C^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = C \end{cases}$$

c) $$(A+C) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$(A+C)^2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(A+C)^3 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} 2 = 2 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2(A+C)$$

$$\Rightarrow \begin{cases} e^A e^C = \left[A^0 + A + \frac{\mathbb{1}}{2!} + \frac{A}{3!} + \frac{\mathbb{1}}{4!} + \cdots \right] \left[C^0 + C + \frac{\mathbb{1}}{2!} + \frac{C}{3!} + \cdots \right] \\ \text{ad} \\ e^{A+C} = \left[(A+C)^0 + (A+C) + \frac{2}{2!} \mathbb{1} + \frac{2}{3!}(A+C) + \frac{4}{5!}(A+C)\mathbb{1} + \cdots \right] \end{cases}$$

$$\Rightarrow e^{A+C} \neq e^A e^C$$

48. Show $Tr(A^{-1}MA) = Tr(M)$

and

$$|A^{-1}MA| = |M|$$

Sol^n

(a) $|A^{-1}MA| = |M|$

$$\underbrace{|A^{-1}||M|}|A| = ?$$

are #s $\rightarrow$ $\underbrace{|A^{-1}||A|}|M| = ?$

$$|A^{-1}A||M| = |\mathbb{1}||M| = |M| \checkmark$$

(b) $Tr(A^{-1}MA) = Tr(M)$

We can permute the order of matrices in a Tr w/o changing the Tr.

$$Tr(A^{-1}MA) = Tr(MAA^{-1}) = Tr(AA^{-1}M)$$

$$Tr(AA^{-1}M) = Tr(\mathbb{1}M)$$

$$= Tr(M) \checkmark$$