NAME ____

Test 2: Physics 410, Mathematical Methods for Physicists
November 26, 2013

- Do 4 of the 6 following problems.
- Mark clearly the questions you wish to have scored.
- Each question is weighted equally.
- The exam is worth 50 points.

Question 1

a. For the complex exponential Fourier series,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{2i\pi nx/P} \tag{1}$$

where P is the period of $f(x)$, derive an expression for the c_n.

b. Given

$$f(x) = \begin{cases} x, & -1 < x < 1 \\ 0, & 1 < |x| < 3 \end{cases} \tag{2}$$

and $f(x)$ has period 6, draw a graph of $f(x)$ between -10 and 10. Is $f(x)$ even, odd, or neither?

c. Find the exponential Fourier series for the $f(x)$ given in part (b).

Question 2

Find the solution for the differential equation

$$(1 + x^2)\,\frac{\partial^2 y}{\partial x} - 2x\,\frac{\partial y}{\partial x} + 2\,y = 0 \tag{3}$$

using the integration by series method, that is, under the assumption that the solution to the above differential equation may be written as

$$\sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+k} \tag{4}$$

where k is a constant as follows:

a. find the indicial equation, and then find the acceptable values for k.

b. Find the two-term recurrence relation for the a_λ.

c. Find the solution for the differential equation.

Question 3

The one-dimensional heat flow equation is given by

$$\kappa^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{5}$$

a. Using separation of variables, find the solution for the one-dimensional heat flow equation.

b. The temperature u of a thin metal rod of length L, extending from $x = 0$ to L, is held at $u = T_\circ = 50°\text{C}$ at $t = 0$. For times $t > 0$, the condition $\frac{\partial u}{\partial x} = 0$ is enforced at the ends of the bar. For these boundary conditions, find $u(x, t)$ as a function of time.

c. What is the temperature at the ends of the bar at times $t = \tau_\circ$ and $5\tau_\circ$, where $\tau_\circ = L^2/(\pi^2\kappa^2)$

d. What is the temperature at the center of the bar at the same times, $t = \tau_\circ$ and $5\tau_\circ$.

Question 4

A box with sides of length $a = 10$ has two insulated sides, one at $y = 0$ and the other at $x = a$, both held at T = 50°C. The other four sides of the box are held at T = 0°C.

a. Find the steady-state temperature inside the box.

b. Estimate the steady-state temperature at the center of the box T(5,5,5).

c. Find the direction energy would flow at the center of the box.

Question 5

The potential on a spherical shell, radius R, is held at

$$\Phi(r = R, \theta) = \Phi_\circ \left(\sin^2\theta + \cos^3\theta\right) \tag{6}$$

a. Express the potential on the spherical shell in terms of Legendre polynomials.

b. Show that the axially symmetric potential $\Phi(r, \theta)$ may be written as

$$\Phi(r, \theta) = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})\ P_l(\theta) \tag{7}$$

by solving the Laplace equation for $\Phi(r, \theta)$ using the method of separation of variables. You may assume that the solution of the Legendre equation for integer l are Legendre polynomials.

c. Find $\Phi(r, \theta)$ inside and outside the spherical shell using the form for $\Phi(r, \theta)$ given in part (b). Note that at ∞, the potential goes to zero and that at the origin, the potential must be well-behaved.

Question 6

We have the function

$$f(x) = \begin{cases} 0, & |t| > T \\ \sin \omega_\circ t, & |t| < T \end{cases} \tag{8}$$

a. Find the Fourier transform of $f(x)$.

b. Show that the larger the value of T, the more the Fourier transform $g(\omega)$ concentrates around the frequencies $\pm\omega_\circ$.