Homework 2

Due: 14 October 2016

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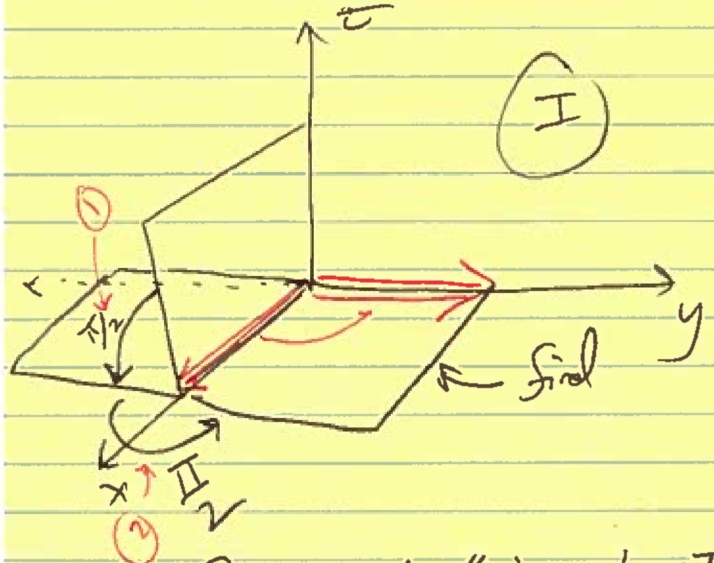
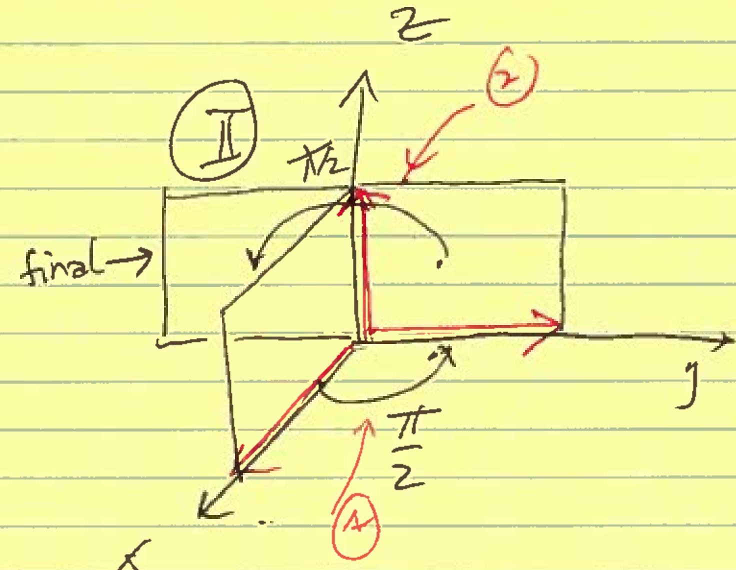
HW 2

LL 3.7(31)





I (a)
$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ +\sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{about } z\text{-axis by } \pi/2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ 0 & +\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}}_{\text{about } x\text{-axis by } \pi/2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ +1 & 0 & 0 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ +x \\ +y \end{pmatrix}}$$

$\Rightarrow$ z rotates into x & x rotates into y
(represents the operation as 1 rotation)

(b) find axis of rotation. Rotate by $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. If we rotate about the vector that points along the axis then the rotation will leave the vector unchanged!

II (a)

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ 0 & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}}_{\text{rotate abt } x\text{-axis by } \pi/2} \underbrace{\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotate abt } z\text{-axis by } \pi/2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -z \\ x \end{pmatrix}}$$

$\Rightarrow$ x rotates into z & z rotates into $-y$

(b)

$$\begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix}$$

(represents operation as one rotation)

$$= \begin{pmatrix} -y_a \\ -z_a \\ x_a \end{pmatrix}$$

$\Rightarrow x_a = -y_a,\ y_a = -z_a,\ x_a = z_a$. Set $x_a = 1$

$\rightarrow x_a = 1,\ y_a = -1,\ z_a = 1$ and the rotation axis points in direction

$$\boxed{\frac{1}{\sqrt{3}}(1, -1, 1)}$$

$\Rightarrow$ rotates $\frac{1}{3}$ of way around $\Rightarrow \frac{2\pi}{3} = 120°$

$$\Rightarrow \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix}$$

$$= \begin{pmatrix} z_a \\ x_a \\ y_a \end{pmatrix}$$

$\Rightarrow x_a = z_a,\ y_a = x_a,\ z_a = y_a$. Set $x_a = 1$

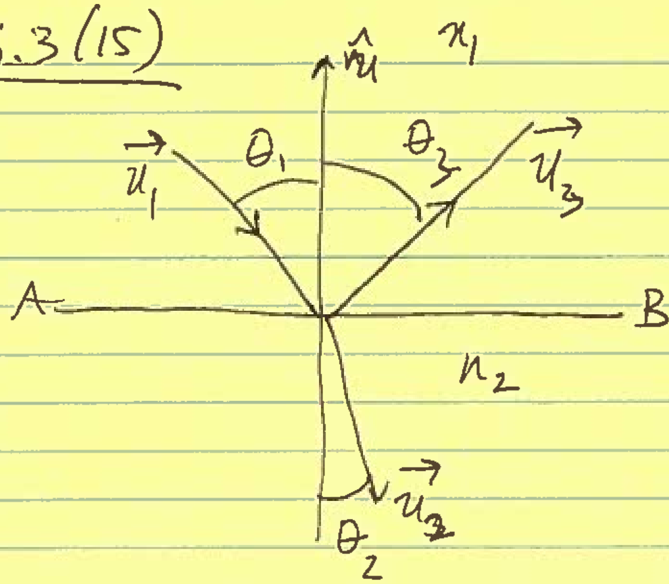
$\Rightarrow x_a = y_a = z_a = 1$ and the unit vector is

$$\boxed{\frac{1}{\sqrt{3}}(1,1,1)}$$

defines the direction of the axis of rotation

$\Rightarrow$ rotates $\frac{1}{3}$ of way around $\Rightarrow \frac{2\pi}{3} = 120°$

12-6.3(15)



(i) $\theta_1 = \theta_3$

(ii) $n_1 \sin\theta_1 = n_2 \sin\theta_2$

Write the above in terms of dot products and cross products

(i) $\theta_1 = \theta_3$ follows from finding the projection of $\vec{u}_1$ & $\vec{u}_3$ onto the normal,

$$\boxed{\hat{n}\cdot\vec{u}_1 + \hat{u}\cdot\vec{u}_3 = 0}$$, where $\vec{u}_1$ & $\vec{u}_3$ are unit vectors

$$\cos(\pi-\theta) + \cos\theta_3 = 0$$

$$-\cos\theta_1 + \cos\theta_3 = 0 \Rightarrow \cos\theta_1 = \cos\theta_3$$

or

$$\theta_1 = \theta_3 \checkmark$$

(ii) Show $n_1 \sin\theta_1$ ~~=~~ $n_2 \sin\theta_2$ as a vector equation b/c of sines suggest that

$$\boxed{n_1 \hat{u}_1 \times \hat{n} - n_2 \hat{u}_2 \times \hat{n} = 0}$$

$$n_1 \sin(\pi-\theta_1) - n_2 \sin(\pi-\theta_2) = 0$$

$$n_1 \sin\theta_1 - n_2 \sin\theta_2 = 0$$

13.6.4(2)

ⓐ $\vec{r}(t) = \hat{\imath}\,t^2 - 2\hat{\jmath}\,t + (t^2+2t)\,\hat{k}$

Show that $\vec{r}(t)$ passes thrugh $(+4,-4,8)$. at what time does it do so?

(i) does $\left.\begin{aligned} t^2 &= 4 \\ -2t &= -4 \\ (t^2+2t) &= 8 \end{aligned}\right\}$ simultaneously?

(ii) from the 2^{nd} formula, we see that $\boxed{t=2}$.

(iii) this is consistent w/ top equation

(iii) $\Rightarrow (2^2+2\times 2) = 8$ ✓

ⓑ find velocity

$$\frac{d\vec{r}}{dt} = 2t\hat{\imath} - 2\hat{\jmath} + (2t+2)\hat{k}$$

$$\Rightarrow \text{speed} = \sqrt{4t^2+4+4+4t^2+8t}$$

$$= \sqrt{8t^2+8t+8}$$

at $\vec{r} = (+4,-4,8)$, $t=2$

$$\Rightarrow \frac{d\vec{r}}{dt} = 4\hat{\imath} - \cancel{4}2\hat{\jmath} + 6\hat{k}$$

and

$$\text{speed} = \sqrt{32+16+8} = \sqrt{56}$$

c) $\vec{r}(t) = (t^2, -2t, t^2+2t)$ w/ tail at origin

(i) $\frac{d\vec{r}}{dt} = (2t, -2, 2t+2)$ is tangent to curve swept out by $\vec{r}(t)$

(ii) $\vec{r}(t)$ passes through $(4,-4,8)$ at $t=2$

$\Rightarrow$ tangent is $(4,-2,6)$ at $(4,-4,8)$

$\Rightarrow$ line has formula,

$$\vec{r}(t) = \vec{r}_0 + \vec{A}t$$

$$= (4,-4,8) + (4,-2,6)t$$

$$\Rightarrow (x-4, y+4, z-8) = (4,-2,6)t$$

$$\Rightarrow \boxed{\frac{x-4}{4} = \frac{y+4}{-2} = \frac{z-8}{6} = t}$$

d) find plane normal to the curve at $(4,-4,8)$

find normal to the curve w/ slope $\vec{A} = (4,-2,6)$

$\cancel{\Rightarrow \vec{A}\cdot\hat{N} = 0 \Rightarrow 4n_x - 4n_y + 8n_z = 0}$

$$\Rightarrow (x-4, y+4, z-8)\cdot\hat{N} = 0$$

$$n_x(x-4) + n_y(y+4) + n_z(z-8) = 0$$

$$4(x-4) - 2(y+4) + 6(z-8) = 0$$

$$4x - 16 - 2y - 8 + 6z - 48 = 0$$

$$\boxed{2x - y + 3z = 36}$$

e) preceding answer (per the text) that $\vec{N} // \vec{A}$. what if $\vec{N} \perp \vec{A}$?

$$\Rightarrow (4,-2,6)\cdot(n_x,n_y,n_z) = 0$$

$$4n_x - 2n_y + 6n_z = 0$$

let $n_x = 1 \Longrightarrow 4 - 2n_y + 6n_z = 0 \Rightarrow n_y = 3n_z + 2$

$$\Rightarrow \vec{N} = \left(1, n_y, \frac{n_y}{3} - \frac{2}{3}\right)$$

and can have an ~~infinite number~~ of planes.

14.6.6(5)

a) find the ~~derivative~~ gradient of $\phi = z\sin y - xz$ at point $(2, \pi/2, -1)$.

$$\vec{\nabla}\phi = \hat{\imath}\frac{\partial}{\partial x}(z\sin y - xz) + \hat{\jmath}\frac{\partial}{\partial y}(z\sin y - xz) + \hat{k}\frac{\partial}{\partial z}(z\sin y - xz)$$

$$\boxed{\vec{\nabla}\phi = \hat{\imath}(-z) + \hat{\jmath}(z\cos y) + \hat{k}(\sin y - x)}$$

at $(2, \frac{\pi}{2}, -1)$

$$\boxed{\vec{\nabla}\phi = \hat{\imath} + \hat{\jmath}(-0) + \hat{k}(-1)}$$

b) from $(2, \pi/2, -1)$, in which direction is ϕ decreasing most rapidly?

$$\vec{\nabla}\phi \downarrow \Rightarrow \boxed{-\hat{\imath} + \hat{k}}$$

c) find derivative of ϕ in the direction $2\hat{\imath} + 3\hat{\jmath}$

$$\vec{\nabla}\phi \cdot (2\hat{\imath} + 3\hat{\jmath})$$

$$= (-z\hat{\imath} + z\cos y\,\hat{\jmath} + (\sin y - x)\hat{k}) \cdot (2\hat{\imath} + 3\hat{\jmath})$$

$$= -2z + 3z\cos y$$

$$= \boxed{-2z\left(1 - \frac{3}{2}\cos y\right)}$$

at $(2, \pi/2, -1) \Rightarrow 2$

15. 6.6(10)

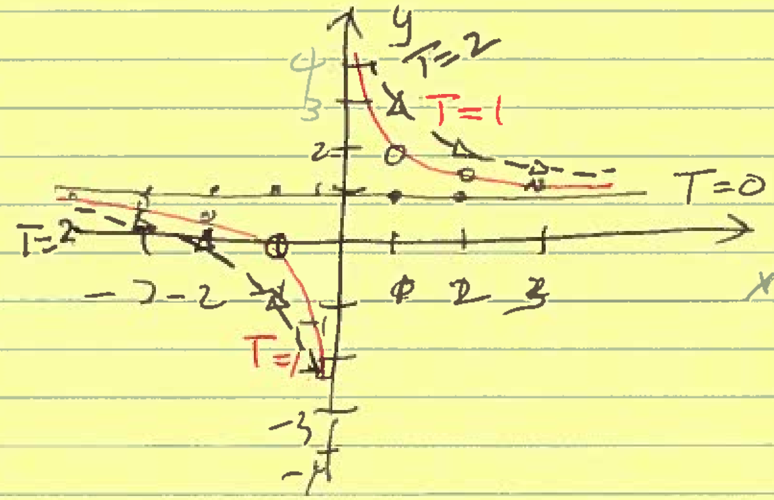
$T(x,y) = xy - x = x(y-1)$

a) Sketch a few isotherms for $T = 0, 1, 2, -1, -2$

$T=0$

x	y
1	1
0	—
−1	1
2	1
−2	1
+3	1
−3	1

$\Rightarrow$



$T=1$

x	y
1	2
−1	0
2	3/2
−2	1/2
3	4/3
−3	1/3

$\Rightarrow$

$T=2$

x	y
1	3
0	$\pm\infty$
−1	−1
2	2
−2	0
3	5/3
−3	1/3

b) find the dir^n in which T changes most rapidly from point (1,1) and the maximum rate of change.

$$T(x,y) = xy - x = x(y-1)$$

find $\vec{\nabla} T = \hat{i}\,[y-1] + \hat{j}\,[x]$

$$\Rightarrow \boxed{\vec{\nabla} T\big|_{(1,1)} = 0\hat{i} + \hat{j}} \Rightarrow \left|\vec{\nabla} T\right|_{(1,1)} = 1$$

c) find $\vec{\nabla} T\big|_{(1,1)} \cdot (3\hat{i} - 4\hat{j})$ from (1,1)

$$= 3(y-1) - 4x \quad \text{at } (1,1)$$

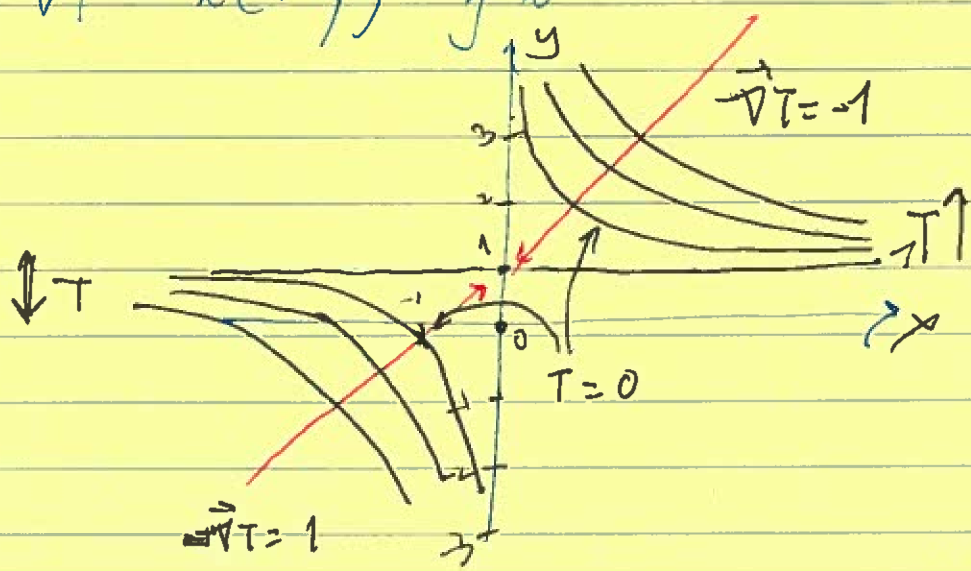
$$= 0 - 4$$

d) Heat flows in direction of $-\vec{\nabla} T$. Sketch a few $-\vec{\nabla} T$ curves.

$$-\vec{\nabla} T = \hat{i}(1-y) - \hat{j}\,x$$

$-\vec{\nabla} T = 1$

x	y
0	1
1	2
2	3
$-\vec{\nabla} T = +1$	
0	1
−1	0
−2	−1
−3	−2



15. 6.6(14)

Suppose a hill has equation, $z = 32 - x^2 - 4y^2$ = height, sketch a contour map. Use $z = 32, 19, 12, 7, 0$

(a) $z=0 \rightarrow 0 = 32 - x^2 - 4y^2 \Rightarrow$

x	y
0	$\pm\sqrt{8}$
± 2	$\pm\sqrt{7}$
± 4	± 2
$\pm\sqrt{32}$	0

(b) $z=32 \Rightarrow 32 = 32 - x^2 - 4y^2$

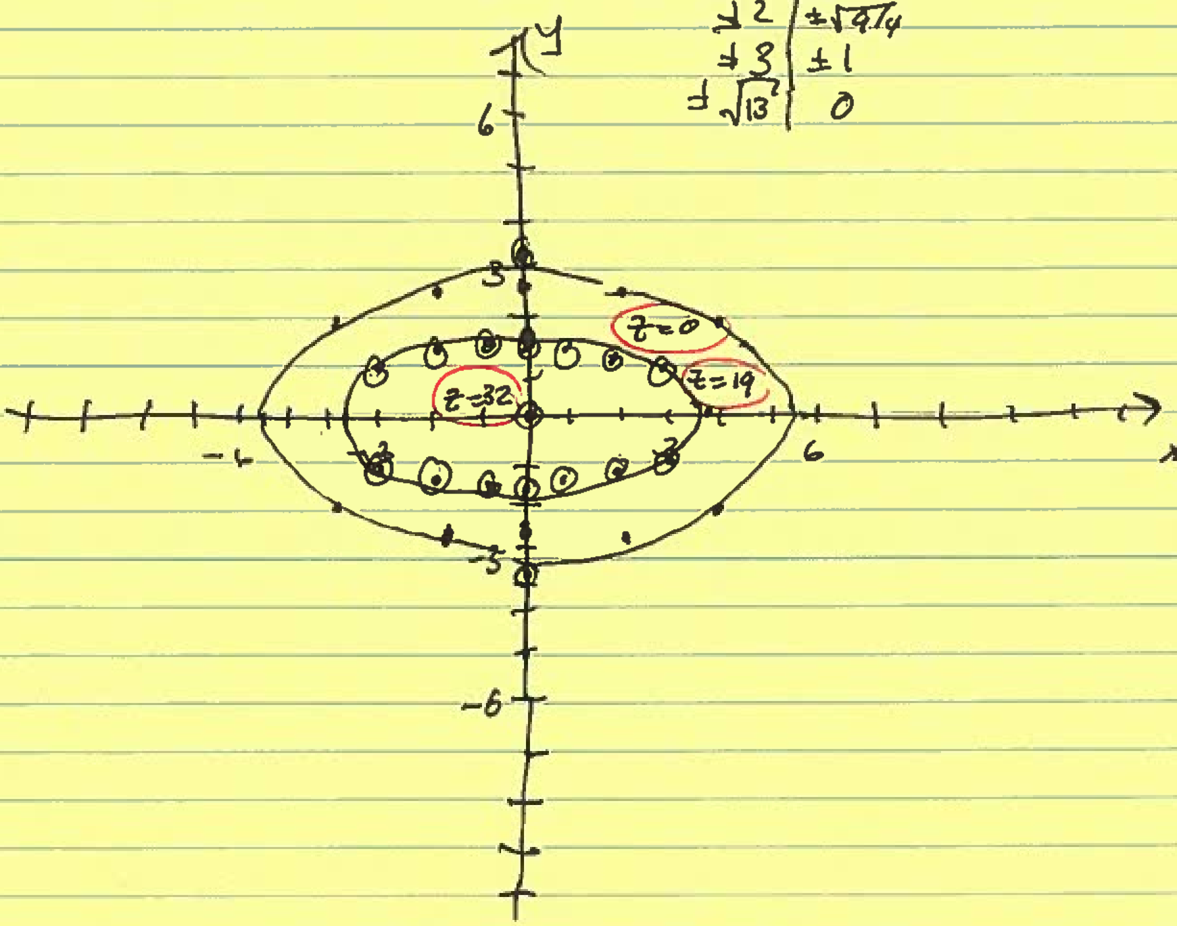
$0 = -x^2 - 4y^2 \Rightarrow x = y = 0$

$\boxed{x^2 = -4y^2}$ X

(c) $z = 19 \Rightarrow 13 = x^2 + 4y^2$

$\Rightarrow$

x	y
0	$\pm\sqrt{13/4}$
± 1	$\pm\sqrt{3}$
± 2	$\pm\sqrt{9/4}$
± 3	± 1
$\pm\sqrt{13}$	0



(b) if you start at (3, 2) in the dir^ (1, 1) are you going uphill or downhill and how fast?

$\vec{\nabla} z = -2x\hat{i} - 8y\hat{j}$ at $(3, 2)$

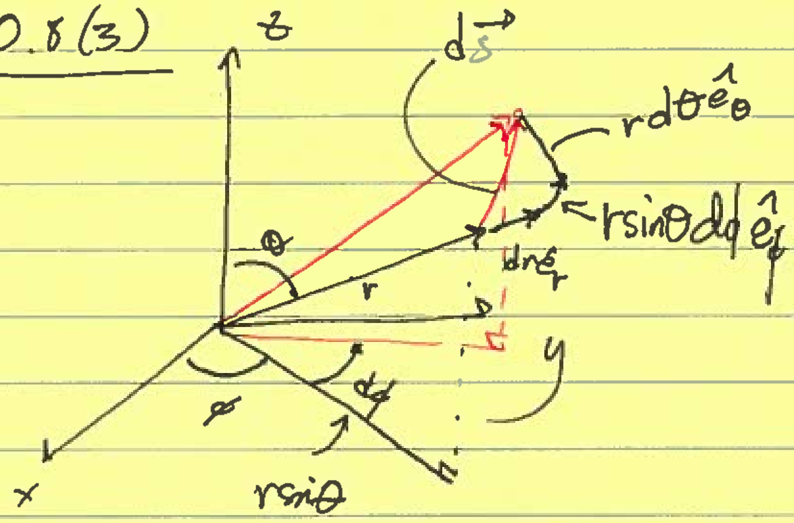
$\rightarrow \boxed{\vec{\nabla} z = -6\hat{i} - 16\hat{j}}$

$\vec{\nabla} z \cdot (1, 1) = -6 - 16 = -22 < 0$

$\rightarrow$ downhill at 22 ft/ft

17. 10.8(3)

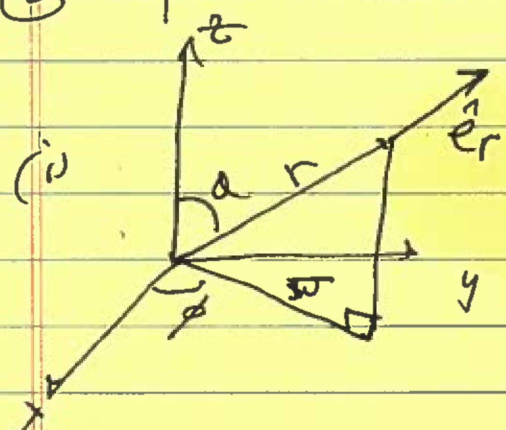
(a)



$$\boxed{\begin{aligned} d\vec{s} &= dr\hat{e}_r + r\sin\theta d\phi\hat{e}_\phi + rd\theta\hat{e}_\theta \\ &\rightarrow ds^2 = d\vec{s}\cdot d\vec{s} = dr^2 + r^2\sin^2\theta\, d\phi^2 + r^2 d\theta^2 \end{aligned}}$$

(b) Express in Cartesian coordinates

(i)



$\hat{e}_r$

(i) project r into xy plane,

$\varpi = r\sin\theta$

(ii) $h_\varpi \hat{e}_\varpi = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j}$

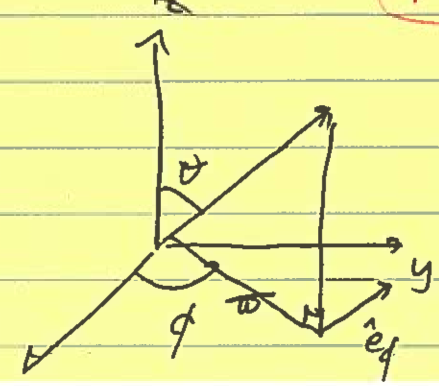
$\hat{e}_r = \hat{e}_\varpi + \hat{z}$

$= \underline{r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}}$

r

$$\boxed{\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}$$

(ii)

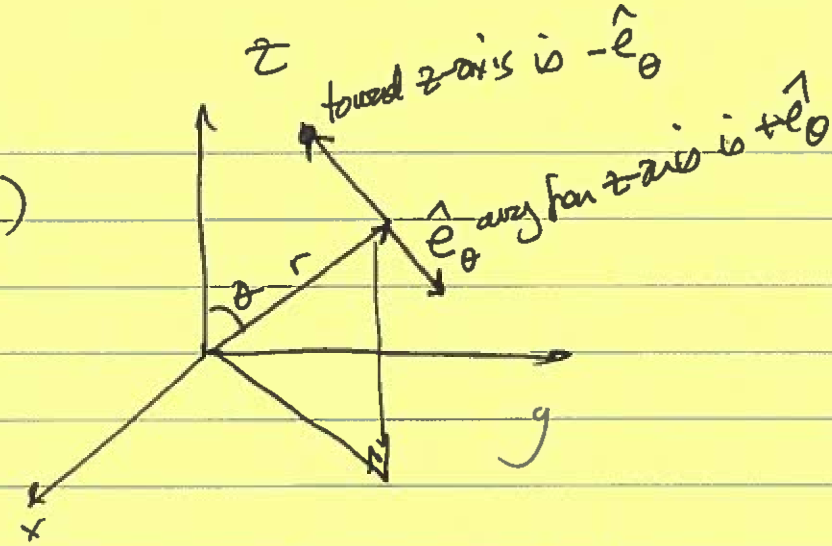


$\hat{e}_\phi$ ① project r into xy plane.

② $h_\phi \hat{e}_\phi = -r\sin\theta\sin\phi\,\hat{i}$

$+ r\sin\theta\cos\phi\,\hat{j}$

(iii)



$$\Rightarrow h_\theta \hat{e}_\theta = r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}$$

$$\boxed{\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}}$$

(c) from a, b, we have $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ in terms of (r,θ,ϕ) & $\hat{i}, \hat{j}, \hat{k}$

and we can find scale factors,

$$\boxed{h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta}$$

(d) from ds^2, we see that the metric is

$$\boxed{g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}}$$

18. 10.8(a)

(a) $\vec{r} = r\hat{e}_r$

$$\frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

$$\frac{d}{dt}\hat{e}_r = \frac{d}{dt}\left[\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right]$$

$$= \left[\cos\theta\dot{\theta}\cos\phi - \sin\theta\sin\phi\dot{\phi},\ \cos\theta\dot{\theta}\sin\phi + \sin\theta\cos\phi\dot{\phi},\ -\sin\theta\dot{\theta}\right]$$

$$= \dot{\theta}\left[\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\right] + \dot{\phi}\left[-\sin\theta\sin\phi, \sin\theta\cos\phi, 0\right]$$

$$= \dot{\theta}\hat{e}_\theta + \dot{\phi}\hat{e}_\phi\sin\theta$$

$$\boxed{\frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\dot{\phi}\hat{e}_\phi\sin\theta}$$

$$d\vec{s} = dr\hat{e}_r + r\sin\theta\, d\phi\,\hat{e}_\phi + r\,d\theta\,\hat{e}_\theta$$

$$\Longrightarrow \boxed{\frac{d\vec{s}}{dt} = \dot{r}\hat{e}_r + r\sin\theta\dot{\phi}\hat{e}_\phi + r\dot{\theta}\hat{e}_\theta}$$

(b) find $\ddot{\vec{r}}$

$$\frac{d}{dt}(\dot{\vec{r}}) = \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \ddot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta$$
$$+ \dot{r}\dot{\phi}\sin\theta\hat{e}_\phi + r\ddot{\phi}\sin\theta\,\hat{e}_\phi + r\dot{\phi}\cos\theta\,\dot{\theta}\,\hat{e}_\phi$$
$$+ \dot{r}\sin\theta\dot{\phi}\dot{\hat{e}}_\phi$$

recall: $\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta + \dot{\phi}\sin\theta\hat{e}_\phi$

~~and also~~

$$\dot{\hat{e}}_\phi = -\cos\phi\dot{\phi}\hat{i} - \sin\phi\dot{\phi}\hat{j} = -\dot{\phi}(\cos\phi\hat{i} + \sin\phi\hat{j})$$

$$\dot{\hat{e}}_\theta = -\sin\theta\dot{\theta}\cos\phi\hat{i} - \cos\theta\sin\phi\dot{\phi}\hat{i} - \sin\theta\dot{\theta}\sin\phi\hat{j}$$
$$+ \cos\theta\cos\phi\dot{\phi}\hat{j} - \cos\theta\dot{\theta}\hat{k}$$
$$= -\dot{\theta}\left[+\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}\right]$$
$$-\dot{\phi}\cos\theta\left[\sin\phi\hat{i} - \cos\phi\hat{j}\right]$$
$$= -\dot{\theta}\hat{e}_r + \dot{\phi}\cos\theta\hat{e}_\phi$$

hmm, what is $-\dot{\phi}\left[\cos\phi\hat{i} + \sin\phi\hat{j}\right]$?

take $\sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta$

$= (\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta) + (\cos^2\theta\cos\phi, \cos^2\theta\sin\phi, -\cos\theta\sin\theta)$

$= (\cos\phi, \sin\phi, 0)$ ✓

$$\frac{d}{dt}\dot{\vec{r}} = \hat{e}_r\left[\ddot{r} + r\dot{\theta}(-\dot{\theta}) + r\sin\theta\dot{\phi}(-\dot{\phi}\sin\theta)\right]$$

$$+\hat{e}_\theta\left[\dot{r}\dot{\theta} + r\ddot{\theta} + \dot{r}(\dot{\theta}) + r\sin\theta\dot{\phi}(-\dot{\phi}\cos\theta)\right]$$

$$+\hat{e}_\phi\left[\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\cos\theta\dot{\theta} + \dot{r}(\dot{\theta}) + r\dot{\theta}(\cos\theta\dot{\phi})\right]$$

Pr. 10.8(6)

find ds^2, scale factors, $d\vec{s}$, d^3x, d^2x, $\vec{a}$ vectors, $\hat{e}$ vectors for parabolic cylinder coordinates, u, v, z

$$x = \frac{1}{2}(u^2 - v^2)$$
$$y = uv$$
$$z = z$$

(a) ds^2 $\quad dx = u\,du - v\,dv$

$$dy = u\,dv + du\,v$$
$$dz = dz$$

$$\Rightarrow ds^2 = (u\,du - v\,dv)^2 + (u\,dv + du\,v)^2 + dz^2$$
$$= du^2(u^2+v^2) + dv^2(v^2+u^2) + dz^2$$
$$- 2uv\,\cancel{du\,dv} + 2\cancel{uv}\,du\,dv$$

$$\boxed{ds^2 = (u^2+v^2)du^2 + (u^2+v^2)dv^2 + dz^2}$$

(b) Scale factors

$$\boxed{h_u = \sqrt{u^2+v^2},\ h_v = \sqrt{u^2+v^2},\ h_z = 1}$$

(c) $d\vec{s}$

$$d\vec{s} = \sqrt{u^2+v^2}\,d\vec{u} + \sqrt{u^2+v^2}\,d\vec{v} + d\vec{z}$$
$$= \underbrace{\sqrt{u^2+v^2}\,du\,\hat{e}_u}_{\hat{a}_u} + \underbrace{\sqrt{u^2+v^2}\,dv\,\hat{e}_v}_{\hat{a}_v} + \underbrace{dz\,\hat{e}_z}_{\hat{a}_z}$$

(d) d^3x, d^2x

$$d^3x = h_u h_v h_z\, du\, dv\, dz$$
$$= (u^2+v^2)\, du\, dv\, dz$$

$$d^2x\,\hat{e}_z = h_u h_v\, du\, dv\, \hat{e}_z = (u^2+v^2)\, du\, dv\, \hat{e}_z$$

$$d^2x\,\hat{e}_u = h_v h_z\, dv\, dz\, \hat{e}_u = \sqrt{u^2+v^2}\, dv\, dz\, \hat{e}_u$$

$$d^2x\,\hat{e}_v = h_u h_z\, du\, dz\, \hat{e}_v = \sqrt{u^2+v^2}\, du\, dz\, \hat{e}_v$$