Homework 3

Due: 21 October 2016, by end of day

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19. page 299, 6.7, Problem 17
20. page 307, 6.8, Problem 16
21. page 308, 6.8, Problem 20
22 page 323, 6.10, Problem 15
23. page 527, 10.9, Problem 1
24. page 528, 10.9, Problem 2

HW #3

19. 6.7(17) Verify b, c, d, g, h, i, j, k

(b) $\vec{\nabla} \times \vec{\nabla}\phi = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}, \frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}, \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)$$

if ϕ & partials of ϕ are continuous

$$\Rightarrow \frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial^2 \phi}{\partial z \partial y}, \ldots$$

$$\rightarrow = (0, 0, 0)$$

(c) $\vec{\nabla}(\vec{\nabla} \cdot \vec{V}) = ?$

(1) $\vec{\nabla} \cdot \vec{V} = \left(\frac{\partial}{\partial x} V_x + \frac{\partial}{\partial y} V_y + \frac{\partial}{\partial z} V_z\right)$

$$\rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) = \hat{i}\left[\frac{\partial^2}{\partial x^2} V_x + \frac{\partial^2}{\partial x \partial y} V_y + \frac{\partial^2}{\partial x \partial z} V_z\right]$$
$$+ \hat{j}\left[\frac{\partial^2}{\partial y \partial x} V_x + \frac{\partial^2}{\partial y^2} V_y + \frac{\partial^2}{\partial y \partial z} V_z\right]$$
$$+ \hat{k}\left[\frac{\partial^2}{\partial z \partial x} V_x + \frac{\partial^2}{\partial z \partial y} V_y + \frac{\partial^2}{\partial z^2} V_z\right]$$
$$= \underbrace{\nabla^2 \vec{V}}_{\checkmark s} + \left(\frac{\partial}{\partial x}\left[\frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right], \frac{\partial}{\partial y}\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z}\right], \frac{\partial}{\partial z}\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right]\right)$$

(f) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = \ ?$

(i) $\vec{\nabla}\times\vec{V} = \left(\frac{\partial}{\partial y}V_z - \frac{\partial}{\partial z}V_y,\ \frac{\partial}{\partial z}V_x - \frac{\partial}{\partial x}V_z,\ \frac{\partial}{\partial x}V_y - \frac{\partial}{\partial y}V_x\right)$

$\rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = \left[\frac{\partial^2}{\partial x\partial y}V_z - \frac{\partial^2}{\partial x\partial z}V_y + \frac{\partial^2}{\partial y\partial z}V_x - \frac{\partial^2}{\partial y\partial x}V_z + \frac{\partial^2}{\partial z\partial x}V_y - \frac{\partial^2}{\partial z\partial y}V_x\right]$

if $\vec{V}$ & partial of $\vec{V}$ are continuous order of differentiation doesn't matter

$= 0$

(g) $\vec{\nabla}\times(\phi\vec{V}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi V_x & \phi V_y & \phi V_z \end{vmatrix} =$

$= \hat{i}\left[\frac{\partial}{\partial y}\phi V_z - \frac{\partial}{\partial z}\phi V_y\right] + \hat{j}\left[\frac{\partial}{\partial z}\phi V_x - \frac{\partial}{\partial x}\phi V_z\right]$

$+ \hat{k}\left[\frac{\partial}{\partial x}\phi V_y - \frac{\partial}{\partial y}\phi V_x\right]$

$= \hat{i}\left[\phi\frac{\partial V_z}{\partial y} + V_z\frac{\partial\phi}{\partial y} - \phi\frac{\partial V_y}{\partial z} - V_y\frac{\partial\phi}{\partial z}\right]$

$+ \hat{j}\left[\phi\frac{\partial V_x}{\partial z} + V_x\frac{\partial\phi}{\partial z} - \phi\frac{\partial V_z}{\partial x} - V_z\frac{\partial\phi}{\partial x}\right]$

$+ \hat{k}\left[\phi\frac{\partial V_y}{\partial x} + V_y\frac{\partial\phi}{\partial x} - \phi\frac{\partial V_x}{\partial y} - V_x\frac{\partial\phi}{\partial y}\right]$

$= \hat{i}\left[\phi\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + V_z\frac{\partial\phi}{\partial y} - V_y\frac{\partial\phi}{\partial z}\right]$

$+ \hat{j}\left[\phi\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + V_x\frac{\partial\phi}{\partial z} - V_z\frac{\partial\phi}{\partial x}\right]$

$+ \hat{k}\left[\phi\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) + V_y\frac{\partial\phi}{\partial x} - V_x\frac{\partial\phi}{\partial y}\right]$

1st column $\qquad\qquad$ $-$2nd column

$$= \phi\, \vec{\nabla}\times\vec{V} - \vec{V}\times\vec{\nabla}\phi$$

(h) $\vec{\nabla}\cdot(\vec{u}\times\vec{v}) = ?$

(i) $\vec{u}\times\vec{v} = (u_y v_z - u_z v_y,\ u_z v_x - u_x v_z,\ u_x v_y - u_y v_x)$

$$\rightarrow \vec{\nabla}\cdot(\vec{u}\times\vec{v}) = \frac{\partial}{\partial x}(u_y v_z - u_z v_y) + \frac{\partial}{\partial y}(u_z v_x - u_x v_z) + \frac{\partial}{\partial z}(u_x v_y - u_y v_x)$$

$$= \left[u_y\frac{\partial v_z}{\partial x} + \frac{\partial u_y}{\partial x}v_z - u_z\frac{\partial v_y}{\partial x} - \frac{\partial u_z}{\partial x}v_y\right]$$

$$+ \left[u_z\frac{\partial v_x}{\partial y} + \frac{\partial u_z}{\partial y}v_x - u_x\frac{\partial v_z}{\partial y} - \frac{\partial u_x}{\partial y}v_z\right]$$

$$+ \left[u_x\frac{\partial v_y}{\partial z} + \frac{\partial u_x}{\partial z}v_y - u_y\frac{\partial v_x}{\partial z} - \frac{\partial u_y}{\partial z}v_x\right]$$

gather terms

$$= u_x\left[-\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right] + u_y\left[\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right] + u_z\left[-\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]$$

$$+ v_x\left[\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right] + v_y\left[-\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right] + v_z\left[\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right]$$

$$= \vec{u}\cdot(-\vec{\nabla}\times\vec{v}) + \vec{v}\cdot(\vec{\nabla}\times\vec{u})$$

(ii) $\vec{\nabla} \times (\vec{u} \times \vec{V})$

(i) $(\vec{u} \times \vec{V}) = (u_y V_z - u_z V_y ,\; u_z V_x - u_x V_z ,\; u_x V_y - u_y V_x)$

$$\to \vec{\nabla} \times (\vec{u} \times \vec{V}) = \hat{i}\left[\frac{\partial}{\partial y}(u_x V_y - u_y V_x) - \frac{\partial}{\partial z}(u_z V_x - u_x V_z)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial z}(u_y V_z - u_z V_y) - \frac{\partial}{\partial x}(u_x V_y - u_y V_x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(u_z V_x - u_x V_z) - \frac{\partial}{\partial y}(u_y V_z - u_z V_y)\right]$$

$$= \hat{i}\Big[\Big(u_x \frac{\partial V_y}{\partial y} + \frac{\partial u_x}{\partial y} V_y - u_y \frac{\partial V_x}{\partial y} - \frac{\partial u_y}{\partial y} V_x\Big)$$
$$- \Big(u_z \frac{\partial V_x}{\partial z} + \frac{\partial u_z}{\partial z} V_x - u_x \frac{\partial V_z}{\partial z} - \frac{\partial u_x}{\partial z} V_z\Big)\Big]$$

$$+ \hat{j}\Big[\Big(u_y \frac{\partial V_z}{\partial z} + \frac{\partial u_y}{\partial z} V_z - u_z \frac{\partial V_y}{\partial z} - \frac{\partial u_z}{\partial z} V_y\Big)$$
$$- \Big(u_x \frac{\partial V_y}{\partial x} + \frac{\partial u_x}{\partial x} V_y - u_y \frac{\partial V_x}{\partial x} - \frac{\partial u_y}{\partial x} V_x\Big)\Big]$$

$$+ \hat{k}\Big[\Big(u_z \frac{\partial V_x}{\partial x} + \frac{\partial u_z}{\partial x} V_x - u_x \frac{\partial V_z}{\partial x} - \frac{\partial u_x}{\partial x} V_z\Big)$$
$$- \Big(u_y \frac{\partial V_z}{\partial y} + \frac{\partial u_y}{\partial y} V_z - u_z \frac{\partial V_y}{\partial y} - \frac{\partial u_z}{\partial y} V_y\Big)\Big]$$

gather terms and add and subtract same terms

$$= \hat{i}\Big[\Big(V_y \frac{\partial}{\partial y} u_x + V_z \frac{\partial u_x}{\partial z} + V_x \frac{\partial}{\partial x} u_x\Big) - V_x \frac{\partial}{\partial x} u_x$$
$$+ \Big(-u_y \frac{\partial}{\partial y} V_x - u_z \frac{\partial}{\partial z} V_x - u_x \frac{\partial}{\partial x} V_x\Big) + u_x \frac{\partial}{\partial x} V_x$$
$$+ V_x \frac{\partial u_x}{\partial x} - u_x \frac{\partial V_x}{\partial x} + \Big(-V_x \frac{\partial u_y}{\partial y} - V_x \frac{\partial u_z}{\partial z} - V_x \frac{\partial u_x}{\partial x}\Big) + \Big(u_x \frac{\partial V_y}{\partial y} + u_x \frac{\partial V_z}{\partial z} + u_x \frac{\partial V_x}{\partial x}\Big)\Big]$$

similarly except for y, z terms

$$+\hat{j}\,[\quad] + \hat{k}\,[\quad]$$

$$= \hat{i}\left[(\vec{V}\cdot\vec{\nabla})u_x + (\vec{u}\cdot\vec{\nabla})V_x - V_x\,\vec{\nabla}\cdot\vec{u} + u_x\,\vec{\nabla}\cdot\vec{V}\right]$$

$$+\hat{j}\,[\quad] + \hat{k}\,[\quad]$$

except for u_y, V_y — except for u_z, V_z

Express as a vector formula $\Rightarrow$

$$\boxed{(\vec{V}\cdot\vec{\nabla})\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{V} - \vec{V}(\vec{\nabla}\cdot\vec{u}) + \vec{u}(\vec{\nabla}\cdot\vec{V})}$$

(j) $\vec{\nabla}(\vec{u}\cdot\vec{V}) = \vec{\nabla}(u_xV_x + u_yV_y + u_zV_z)$

look at x-cmpt

$$= \left[u_x\frac{\partial V_x}{\partial x} + \frac{\partial u_x}{\partial x}V_x + u_y\frac{\partial V_y}{\partial x} + \frac{\partial u_y}{\partial x}V_y + V_z\frac{\partial V_z}{\partial x} + \frac{\partial u_z}{\partial x}V_z\right]$$

$$= \Big[\underbrace{u_x\frac{\partial}{\partial x}V_x + u_y\frac{\partial}{\partial y}V_x + u_z\frac{\partial}{\partial z}V_x}_{(\vec{u}\cdot\vec{\nabla})V_x} + \underbrace{V_x\frac{\partial}{\partial x}u_x + V_y\frac{\partial}{\partial y}u_x + V_z\frac{\partial}{\partial z}u_x}_{(\vec{V}\cdot\vec{\nabla})u_x}\Big]$$

$$+\Big[+u_y\underbrace{\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)}_{\vec{\nabla}\times\vec{V}|_z} + V_y\underbrace{\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)}_{\vec{\nabla}\times\vec{u}|_z}\Big]$$

$$+\Big[u_z\underbrace{\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)}_{-\vec{\nabla}\times\vec{V}|_y} + V_z\underbrace{\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right)}_{-\vec{\nabla}\times\vec{u}|_y}\Big]$$

$$= (\vec{u}\cdot\vec{\nabla})V_x + (\vec{V}\cdot\vec{\nabla})u_x + \vec{u}\times(\vec{\nabla}\times\vec{V})\Big|_{x\text{-compnt}}$$

$$+ \vec{V}\times(\vec{\nabla}\times\vec{u})\Big|_{x\,\text{compnt}}$$

similarly for y, z compnts

$$\Longrightarrow \boxed{= (\vec{u}\cdot\vec{\nabla})\vec{V} + (\vec{V}\cdot\vec{\nabla})\vec{u} + \vec{u}\times(\vec{\nabla}\times\vec{V}) + \vec{V}\times(\vec{\nabla}\times\vec{u})}$$

c $\vec{\nabla}\cdot(\vec{\nabla}\phi\times\vec{\nabla}\psi) = ?$

(i) $$\vec{\nabla}\phi\times\vec{\nabla}\psi = \left(\frac{\partial}{\partial y}\phi\frac{\partial}{\partial z}\psi - \frac{\partial}{\partial z}\phi\frac{\partial}{\partial y}\psi,\ \frac{\partial}{\partial z}\phi\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}\phi\frac{\partial\psi}{\partial z},\ \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x}\right)$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\phi\times\vec{\nabla}\phi) = \left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}\right)\right]$$

$$= 0$$

20 . 6.8(16) Given $\vec{F}_1 = 2x\hat{i} - 2yz\hat{j} - y^2\hat{k}$

ϕ

$\vec{F}_2 = y\hat{i} - x\hat{j}$

a) Are these conservative? If conservative find the potential.

(i) $$\vec{\nabla}\times\vec{F}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -2yz & -y^2 \end{vmatrix} = (-2y+2y, 0-0, 0-0)$$

$= 0 \rightarrow$ conservative

$$\vec{F}_1 = \vec{\nabla}\phi = ? = \left(\frac{\partial}{\partial x}\phi, \frac{\partial}{\partial y}\phi, \frac{\partial}{\partial z}\phi\right) = (2x, -2yz, -y^2)$$

Integrate $\xrightarrow{\text{yields}}$

$\phi = x^2 + f(y,z)$ — $\frac{\partial\phi}{\partial x}$ term

$= -y^2z + g(x,z)$ — $\frac{\partial\phi}{\partial y}$ term

$= -y^2z + h(x,y)$ — $\frac{\partial\phi}{\partial z}$ term

functions of (y,z), (x,z) & (x,y)

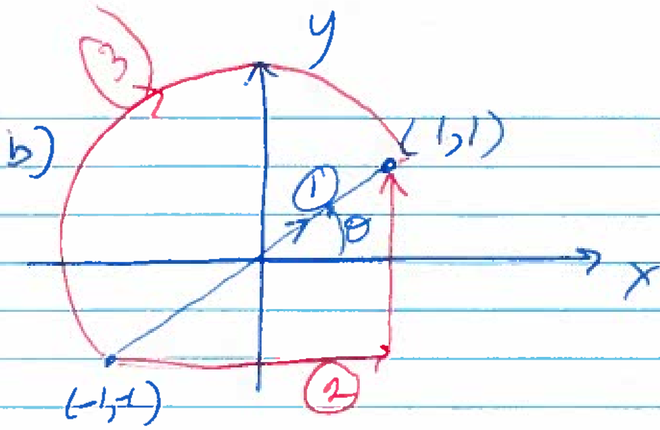
$$\Longrightarrow \boxed{\phi = -y^2z + x^2 + C}$$

(ii) $$\vec{\nabla}\times\vec{F}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = (0-0, 0-0, \underbrace{-1-1}_{-2})$$

$\neq 0$

$\Longrightarrow$ not conservative

b)



Integrate $\int \vec{F}_2 \cdot d\vec{r}$ along the 3 paths shown at left

$$② \int \vec{F}_2 \cdot d\vec{r} = \int_{(-1,-1)}^{(1,-1)} y\,dx - \int_{(1,-1)}^{(1,1)} x\,dy$$

$$= \int_{-1}^{1} (-1)\,dx - \int_{-1}^{1} (1)\,dy$$

$$\boxed{\int \vec{F}_2 \cdot d\vec{r} = -2 - (2) = -4}$$

$$① \int_{①} \vec{F}_2 \cdot d\vec{r} = \int_{①} (y, -x) \cdot d\vec{r}$$

$$= \int_{①} (y\,dx - x\,dy)$$

$$= \int_{①} (x\,dx - x\,[dx])$$

, $y = x$ is path and

$$dr^2 = dx^2 + dy^2$$

$$= dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right) = 2dx^2$$

$$\boxed{\int \vec{F}_2 \cdot d\vec{r} = 0}$$

$$③ \int_{③} \vec{F}_2 \cdot d\vec{r} = \int_{③} (y, -x) \cdot d\vec{r}$$

, $\overbrace{x^2 + y^2}^{r^2} = 2$ is path and

$$= \int \left(r\sin\theta, -r\cos\theta\right) \cdot \left(\hat{x}\,dx + \hat{j}\,dy\right)$$

let's use polar coordinates

$$dr^2 = r_0^2\,d\theta^2$$

$$\overset{③}{=} \int [r\sin\theta\,dx - r\cos\theta\,dy]$$

$r = $ constant

(i) $x = r\cos\theta$

$dx = -r\sin\theta\,d\theta$

$y = r\sin\theta$

$dy = r\cos\theta\,d\theta$

$$= \int \left[r\sin\theta(-r\sin\theta)\,d\theta - r\cos\theta(r\cos\theta)\,d\theta \right]$$

$$= -\int r^2(\sin^2\theta + \cos^2\theta)\,d\theta$$

$$= -r^2(\theta_1 - \theta_0)$$

where $\cos\theta_0 = \cos(\theta_1 + \pi)$ $\quad \cos\theta_1 = \frac{1}{\sqrt{2}}$

$$= -\cos\theta_1$$

$$= -\frac{1}{\sqrt{2}}$$

$$\Rightarrow -r^2\left(\frac{\pi}{4} - \left[+\frac{\pi}{4} + \pi\right]\right) = -r^2\left[-\pi\right] = 2\pi$$

$$\boxed{\int_{③} \vec{F}_2 \cdot d\vec{r} = 2\pi}$$

21. 6.8(20) $\vec{F}_{on} = -mg\hat{k}$ for $h \ll R_*$

$= -\frac{c}{r^3}\vec{r}$ for ~~$h \ll R_*$~~ arbitrary h

① $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -mg \end{vmatrix} = (0-0, 0-0, 0-0) = 0$

$\rightarrow$ conservative

$\vec{F} = -\vec{\nabla}\Phi$ and $F_x = F_y = 0 \rightarrow$ indept of x, y

$F_z = -mg = -\frac{\partial\Phi}{\partial z} \Rightarrow \boxed{\Phi = mgz + c_0}$

② $\vec{\nabla}\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ -\frac{c}{r^3} & 0 & 0 \end{vmatrix}$

$= \frac{1}{r^2\sin\theta}(0-0, 0-0, 0-0) = 0$

$\rightarrow$ conservative

$-\int d\Phi = \int \vec{F}\cdot d\vec{r} \Rightarrow -\int d\Phi = \int -\frac{c}{r^3}\vec{r}\cdot d\vec{r}$

integrate from $r=\infty$ to r $\rightarrow$

$-\Phi(r) + \Phi(\infty) = -\int_\infty^r \frac{c}{r^2}dr$

$\boxed{\Phi(r) = -\frac{c}{r}\Big|_\infty^r = -\frac{c}{r}}$

b/c $\Phi(\infty) \rightarrow 0$

22. 6.10 (15)

Given $\rho(x,y,z,t)$. If we follow a streamline then (x,y,z) are functions of time such that the flow velocity is

$$\vec{v} = \hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}$$

a) Show that

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\vec{v}\cdot\vec{\nabla})\rho$$

(1) $$\frac{d}{dt}\rho(x,y,z,t) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}v_x + \frac{\partial \rho}{\partial y}v_y + \frac{\partial \rho}{\partial z}v_z$$

$$\boxed{\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\vec{v}\cdot\vec{\nabla})\rho}$$

b) Equation (10.9) $\vec{\nabla}\cdot\vec{v}\rho + \frac{\partial \rho}{\partial t} = 0$

$$\Rightarrow \frac{d\rho}{dt} = -\vec{\nabla}\cdot\vec{v}\rho + (\vec{v}\cdot\vec{\nabla})\rho$$

$$= -(\vec{v}\cdot\vec{\nabla})\rho - \rho(\vec{\nabla}\cdot\vec{v}) + (\vec{v}\cdot\vec{\nabla})\rho$$

$$\Rightarrow \frac{d\rho}{dt} + (\vec{\nabla}\cdot\vec{v})\rho = 0$$

$$\frac{d\rho}{dt} = (\vec{v}\cdot\vec{\nabla})\rho \overbrace{- \vec{\nabla}\cdot(\rho\vec{v})}^{\text{substitute for } \frac{\partial\rho}{\partial t}}$$

$$= (\vec{v}\cdot\vec{\nabla})\rho - \rho(\vec{\nabla}\cdot\vec{v}) - (\vec{v}\cdot\vec{\nabla})\rho$$

$$= -\rho(\vec{\nabla}\cdot\vec{v})$$

$$\Longrightarrow \boxed{\frac{d\rho}{dt} + \rho(\vec{\nabla}\cdot\vec{v}) = 0}$$

23. 10.9(1)

Prove 9.4, $\vec{\nabla}\cdot\left(\frac{\hat{e}_3}{h_1 h_2}\right) = \vec{\nabla}\cdot\left(\frac{\hat{e}_2}{h_1 h_3}\right) = \vec{\nabla}\cdot\left(\frac{\hat{e}_3}{h_1 h_2}\right) = 0$

Sol^{n}

okay $\vec{\nabla} = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial x_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial x_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial x_3}$ (9.2)

(i) find $\vec{\nabla} x_1 = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial x_1} x_1 = \frac{\hat{e}_1}{h_1}$, $\frac{\partial x_1}{\partial x_2} = \frac{\partial x_1}{\partial x_3} = 0$

$\rightarrow \vec{\nabla} x_2 = \frac{\hat{e}_2}{h_2}$ and $\vec{\nabla} x_3 = \frac{\hat{e}_3}{h_3}$

(ii) let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ form a right-handed ~~tetrad~~ triad

so that

$$(\vec{\nabla} x_1) \times (\vec{\nabla} x_2) = \left(\frac{\hat{e}_1}{h_1}\right) \times \left(\frac{\hat{e}_2}{h_2}\right)$$

$$= \frac{1}{h_1 h_2}\hat{e}_3$$

(iii) take $\vec{\nabla}\cdot\left[\vec{\nabla} x_1 \times \vec{\nabla} x_2\right] = 0$ (ID ~~8~~)

$$\cancel{= \vec{\nabla} x_1 \cdot (-\vec{\nabla} \times \vec{\nabla} x_2) + \vec{\nabla} x_2 \cdot (\vec{\nabla} \times \vec{\nabla} x_1)}$$

$$\cancel{= \frac{\hat{e}_1}{h_1 h_2}\cdot\left(-\vec{\nabla}\times\frac{\hat{e}_2}{h_2}\right) + \frac{\hat{e}_2}{h_2}\cdot\left(\vec{\nabla}\times\frac{\hat{e}_1}{h_1}\right)}$$

$$= \vec{\nabla}\cdot\left[\frac{\hat{e}_3}{h_1 h_2}\right]$$

$$= 0$$

Repeat for $\vec{\nabla} x_3 \times \vec{\nabla} x_1 = \frac{\hat{e}_2}{h_1 h_3}$

ad

$\vec{\nabla} x_2 \times \vec{\nabla} x_3 = \frac{\hat{e}_1}{h_2 h_3}$

24. 10.9(2)

Derive 9.11, $\vec{\nabla} \times \vec{V} = \dfrac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$

Sol^{n}

(i) recall $\vec{\nabla} x_1 = \dfrac{\hat{e}_1}{h_1}$ and then take

$$\vec{\nabla} \times (\vec{\nabla} x_1) = \vec{\nabla} \times \frac{\hat{e}_1}{h_1} \checkmark = 0 \quad (\text{IB } 6)$$

(ii) write $\vec{V} = \dfrac{\hat{e}_1}{h_1}(h_1 V_1) + \dfrac{\hat{e}_2}{h_2}(h_2 V_2) + \dfrac{\hat{e}_3}{h_3}(h_3 V_3)$

(iii) take $\vec{\nabla} \times \vec{V} = \vec{\nabla} \times \left(\dfrac{\hat{e}_1}{h_1} h_1 V_1\right) + \vec{\nabla} \times \left(\dfrac{\hat{e}_2}{h_2} h_2 V_2\right)$

$$+ \vec{\nabla} \times \left(\frac{\hat{e}_3}{h_3} h_3 V_3\right)$$

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$$= \Big(h_1 V_1 \cancel{\vec{\nabla} \times \frac{\hat{e}_1}{h_1}}^{\,0, \text{ from (i) above}} - \Big(\frac{\hat{e}_1}{h_1} \times \vec{\nabla}\Big) h_1 V_1\Big)$$

$$+ \Big(h_2 V_2 \cancel{\vec{\nabla} \times \frac{\hat{e}_2}{h_2}}^{\,0} - \Big(\frac{\hat{e}_2}{h_2} \times \vec{\nabla}\Big) h_2 V_2\Big)$$

$$+ \Big(h_3 V_3 \cancel{\vec{\nabla} \times \frac{\hat{e}_3}{h_3}}^{\,0} - \Big(\frac{\hat{e}_3}{h_3} \times \vec{\nabla}\Big) h_3 V_3\Big)$$

$$= -\left(0, -\frac{1}{h_1 h_3}\frac{\partial}{\partial x_3} h_1 V_1, \frac{1}{h_1}\frac{1}{h_2}\frac{\partial}{\partial x_2} h_1 V_1\right)$$

$$-\left(\frac{1}{h_2}\frac{1}{h_3}\frac{\partial}{\partial x_3} h_2 V_2, 0, -\frac{1}{h_2}\frac{1}{h_1}\frac{\partial}{\partial x_1} h_2 V_2\right)$$

$$-\left(-\frac{1}{h_3}\frac{1}{h_2}\frac{\partial}{\partial x_2} h_3 V_3, \frac{1}{h_3}\frac{1}{h_1}\frac{\partial}{\partial x_1} h_3 V_3, 0\right)$$

$$= \left(\frac{1}{h_2 h_3}\frac{\partial}{\partial x_2} h_3 V_3 - \frac{1}{h_2 h_3}\frac{\partial}{\partial x_3} h_2 V_2, \frac{1}{h_1 h_3}\frac{\partial}{\partial x_3} h_1 V_1 - \frac{1}{h_1 h_3}\frac{\partial}{\partial x_1} h_3 V_3,\right.$$

$$\left.\frac{1}{h_1 h_2}\frac{\partial}{\partial x_1} h_2 V_2 - \frac{1}{h_1 h_2}\frac{\partial}{\partial x_2} h_1 V_1\right)$$

$$= \frac{1}{h_1 h_2 h_3}\left(h_1 \frac{\partial}{\partial x_2} h_3 V_3 - h_1 \frac{\partial}{\partial x_3} h_2 V_2,\right.$$

$$h_2 \frac{\partial}{\partial x_3} h_1 V_1 - h_2 \frac{\partial}{\partial x_1} h_3 V_3,$$

$$\left. h_3 \frac{\partial}{\partial x_1} h_2 V_2 - h_3 \frac{\partial}{\partial x_2} h_1 V_1\right)$$

$$= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix} \quad ✓$$