Homework 7

Due: 24 November 2016

The problems are relevant to Test 2; they are further examples of problems solved by Separation of Variables and Fourier series, and considerations of the d'Alembert formulation. The due date is meant to emphasize that these types of questions may appear on Test 2 (the problem set may be turned in in class on Tuesday following Thanksgiving break.

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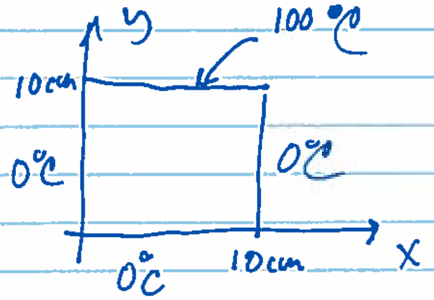
42. page 626, 13.2.10
43. page 626, 13.2.14
44. page 632, 13.3.7
45. page 637, 13.4.2
46. (a) Discuss the possibility of extending the d'Alembert solution to the two-dimensional equation wave equation. (b) Discuss the possibility of solutions of the form $u(x,t) = e^{\lambda x+\mu y}$ for the equation,

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = h$$

where $a, b, c, d, e, f,$ and h are constants.

HW#7

42. 13.2.10



$$\nabla^2 T = 0$$

(i) $\frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T = 0$

let $T(x,y) = f(x) g(y)$

$$\Rightarrow g(y) \frac{\partial^2 f(x)}{\partial x^2} + f(x) \frac{\partial^2 g(y)}{\partial y^2} = 0$$

$$\rightarrow \underbrace{\frac{1}{f(x)} \frac{\partial^2 f(x)}{\partial x^2}}_{-k^2} + \underbrace{\frac{1}{g(y)} \frac{\partial^2 g(y)}{\partial y^2}}_{+k^2} = 0$$

$\Rightarrow$ a) $\frac{1}{g}\frac{\partial^2 g}{\partial y^2} = k^2 \rightarrow \frac{\partial^2 g}{\partial y^2} - k^2 g = 0 \rightarrow g = g_0 e^{-ky} + g_1 e^{+ky}$

b) $\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = -k^2 \rightarrow \frac{\partial^2 f}{\partial x^2} + k^2 f = 0 \rightarrow f = f_0 \cos kx + f_1 \sin kx$

and $\boxed{T(x,y) = (f_0 \cos kx + f_1 \sin kx)(g_0 e^{-ky} + g_1 e^{+ky})}$

(II) BCs, $\underbrace{x = 0, 10\text{cm}, \; T(x,y) = 0}_{①}$, $y = 0, \; T(x,y) = 0$

$\underbrace{y = 10\text{cm}, \; T(x,y) = 100°C}_{③}$

$\Rightarrow (x=0, T=0) \rightarrow \underline{f_0 = 0}$ & $k10 = n\pi$

$\underline{k = \frac{n\pi}{10}}$

and

$\Rightarrow (y=0, T=0) \rightarrow \underline{g_0 = -g_1} \Rightarrow \underline{2g_1 \sinh ky}$ is solution for y

and

$\Rightarrow (y=0, T=100) \Rightarrow 100 = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{10}x\right) \sinh\left(\frac{n\pi}{10}y\right), y=10$

$$100 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{10}x\right)\sinh(n\pi)$$

find C_n

$$\int_0^{10} 100 \sin\left(\frac{n'\pi}{10}x\right)dx = \sum_{n=1}^{\infty} C_n \sinh(n\pi)\int_0^{10} \sin\left(\frac{n\pi}{10}x\right)dx \sin\left(\frac{n'\pi}{10}x\right)$$

$$-100\left(\frac{10}{n'\pi}\right)\cos\left(\frac{n'\pi}{10}x\right)\Big|_0^{10} = C_n \sinh n\pi \left(\frac{10}{2}\right)\delta_{nn'}$$

$$-100\left(\frac{10}{n'\pi}\right)\Big[\underbrace{\cos n'\pi - 1}_{0,\, n' \text{ even}}\Big] = 5\sinh n\pi\, C_n \delta_{nn'}$$

$$200\left(\frac{10}{n'\pi}\right), \; n \text{ odd} = 5\sinh n\pi\, C_n \delta_{nn'}$$

$$\Longrightarrow \; C_n = \frac{400}{n\pi \sinh n\pi}, \; n \text{ odd}$$

and

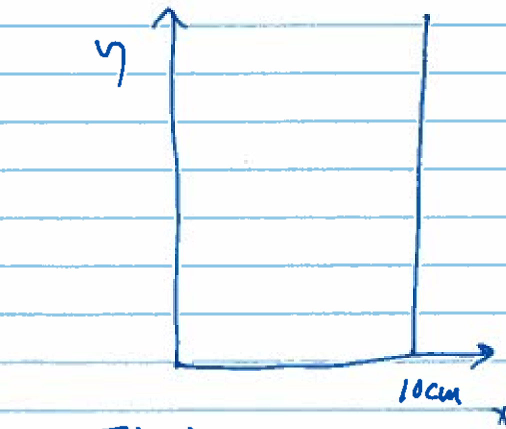
$$T(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{400}{n\pi \sinh n\pi} \sin\left(\frac{n\pi}{10}x\right)\sinh\left(\frac{n\pi}{10}y\right)$$

III $$T(5,5) = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{400}{n\pi \sinh n\pi}\left[\sin\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{400}{\pi \sinh \pi}\sinh\left(\frac{\pi}{2}\right) - \frac{400}{3\pi \sinh 3\pi}\left[\sinh\frac{3\pi}{2}\right]$$

$$+ \frac{400}{5\pi \sinh 5\pi}\left[\sinh\frac{5\pi}{2}\right] - \cdots$$

43. 13.2.14



$$\boxed{\nabla^2 T = 0}$$

fom 13.2.10

$$\boxed{T(x,y) = (f_0 \cos kx + f_1 \sin kx)(g_0 e^{-ky} + g_1 e^{ky})}$$

a) $T(x,0) = x-5$, $\frac{\partial T}{\partial x} = 0$ at $y = 0, 10$, $\underbrace{T(x,y) = 0 \text{ as } y \to \infty}_{\to g_1 = 0}$

b) and $\frac{\partial T}{\partial x} = (-k f_0 \sin kx + k f_1 \cos kx) g_0 e^{-ky}$

at $x = 0, 10$, $\frac{\partial T}{\partial x} = 0 \longrightarrow f_1 = 0$ and $k10 = n\pi$

c) $$\boxed{T(x,y) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{10} x\right) e^{-\left(\frac{n\pi}{10} y\right)}}$$

d) at $y = 0$, $T(x,0) = x - 5$

$$\Rightarrow (x-5) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{10} x\right)$$

(i) $10 C_0 = \int_0^{10} (x-5)\,dx = \frac{x^2}{2} - 5x \Big|_0^{10} = 50 - 50 = 0 \rightarrow C_0 = 0$

(ii) $$\int_0^{10} (x-5) \cos\left(\frac{n'\pi}{10} x\right) dx = \sum_{n=0}^{\infty} C_n \int_0^{10} \cos\left(\frac{n\pi}{10} x\right) \cos\left(\frac{n'\pi}{10} x\right) dx$$

$$\left\{ x \frac{10}{n'\pi} \sin\left(\frac{n'\pi}{10} x\right)\Big|_0^{10} - \frac{10}{n'\pi} \int_0^{10} \sin\left(\frac{n'\pi}{10} x\right) dx - 5 \frac{10}{n'\pi} \sin\left(\frac{n'\pi}{10} x\right)\Big|_0^{10} \right\} = C_n\, 5\, \delta_{nn'}$$

$$5 c_n \delta_{nn'} = \frac{10}{n'\pi}\left[0 + \cos\left(\frac{n'\pi}{10}x\right)\Big|_0^{10} \frac{10}{n'\pi}\right]$$

$$- \frac{50}{n'\pi}\left[0\right]$$

$$= \frac{100}{(n\pi)^2}\left(\cos n\pi - 1\right), \text{ n even} \rightarrow C_n = 0$$

$$, \text{ n odd} \rightarrow C_n = -\frac{40}{(n\pi)^2}$$

$$\Rightarrow \boxed{T(x,y) = -\sum_{n=1}^{\infty} \frac{40}{n^2\pi^2} \cos\left(\frac{n\pi}{10}x\right) e^{-\left(\frac{n\pi}{10}y\right)}}$$

e) Suppose $T = f(x) = x$ at $y = 0$, find $T(x,y)$

$$\rightarrow 10 C_0 = \int_0^{10} x\,dx = 50 \rightarrow \boxed{C_0 = 5}$$

f) $$C_n 5 \delta_{nn'} = \int_0^{10} x \cos\left(\frac{n'\pi}{10}x\right) dx$$

$$= x \sin\left(\frac{n'\pi}{10}x\right)\frac{1}{n'\pi}\Big|_0^{10} - \frac{1}{n'\pi}\int_0^{10} \sin\left(\frac{n'\pi}{10}x\right) dx$$

$$= 0 + \frac{1}{n'^2\pi^2}\cos\left(\frac{n\pi}{10}x\right)\Big|_0^{10}$$

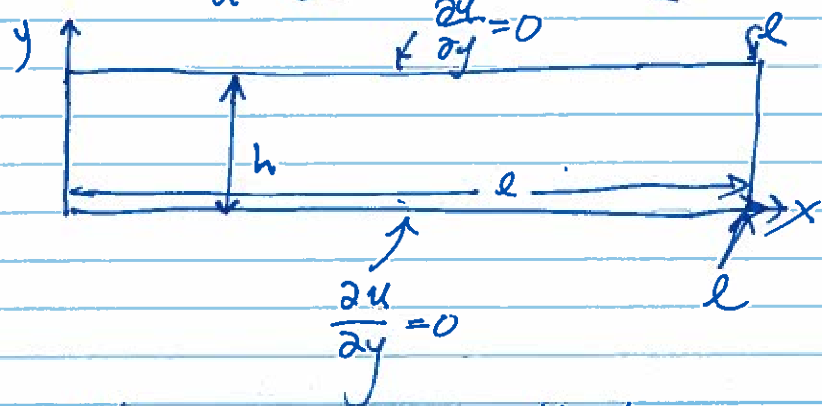
$$5 \delta_{nn'} C_n = \frac{1}{n'^2\pi^2}\left[\cos n'\pi - 1\right]$$

$$C_n = \frac{1}{5n^2\pi^2}(-2), \text{ n odd}$$

$$\rightarrow \boxed{T(x,y) = 5^\circ C - \sum_{n=1}^{\infty} \frac{2}{5n^2\pi^2} \cos\left(\frac{n\pi}{10}x\right) e^{-\frac{n\pi}{10}y}}$$

44. B.3.7

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$ diffusion equation



a) at $t=0$, the ends of the bar at $x \to l$ are also held at $\frac{\partial T}{\partial x} = 0$

find $T(x,y,t)$

b) Initially $T(x,y) = x$

c) $\nabla^2 T = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$; let $T(x,y,t) = f(x)g(y)\tau(t)$

$$\Rightarrow g\tau\frac{\partial^2 f}{\partial x^2} + f\tau\frac{\partial^2 g}{\partial y^2} = \frac{1}{\alpha^2} fg\frac{\partial \tau}{\partial t}$$

(i) $\rightarrow \frac{1}{f}\frac{\partial^2 f}{\partial x^2} + \frac{1}{g}\frac{\partial^2 g}{\partial y^2} = \underbrace{\frac{1}{\alpha^2 \tau}\frac{\partial \tau}{\partial t}} = -k^2$

$$\hookrightarrow \ln\tau\Big|_{t_0}^{t} = -k^2\alpha^2 t\Big|_{t_0}^{t}$$

$$\tau(t) = \tau_0 e^{-k^2\alpha^2(t-t_0)}$$

(ii) $\underbrace{\frac{1}{f}\frac{\partial^2 f}{\partial x^2} + \frac{1}{g}\frac{\partial^2 g}{\partial y^2}} = -k^2$

$$-\alpha^2 - \beta^2 = -k^2$$

$$\rightarrow \frac{1}{f}\frac{\partial^2 f}{\partial x^2} = -\gamma^2, \quad \frac{1}{g}\frac{\partial^2 g}{\partial y^2} = -\beta^2 \quad \text{where } \boxed{\gamma^2+\beta^2=k^2}$$

$$\boxed{f=(f_0\cos\gamma x + f_1\sin\gamma x), \quad g=(g_0\cos\beta y + g_1\sin\beta y)}$$

(iii) for $t=0$ and beyond $\frac{\partial f}{\partial x}=0$ at $x=0,\ell$

and

$\frac{\partial g}{\partial y}=0$ at $y=0,h$

$\Rightarrow f_1=0,\ \gamma=\frac{n\pi}{\ell}$ $\quad\Rightarrow g_1=0,\ \beta=\frac{n\pi}{h}$

$$\Rightarrow T(x,y,t) = f_0\cos\frac{n\pi}{\ell}x\, g_0\cos\left(\frac{n\pi}{h}y\right)_1 e^{-k^2\alpha^2 t}, \quad \left(\frac{t_0=0}{\text{assume}}\right)$$

d) Initial state: $T(x,y) = x$; b/c $\nabla^2 T=0$ when $\frac{\partial T}{\partial t}=0$

$\Rightarrow T(x,y)=x$ is indeed a steady sol^{n} to Laplace equation

e) final state: $T(x,y)=T_0$ b/c $\frac{\partial T}{\partial x}=0$ at $x=0,\ell$

$\Rightarrow T=$ constant in bar is consistent w/ BCs and $\nabla^2 T=0$

f) find C_n where solution is

$$f(x,y,t) = T_0 + \sum_{\substack{n=0,\\ m=1}}^{\infty} C_{nm} \cos\left(\frac{n\pi}{\ell}x\right) \cos\left(\frac{m\pi}{h}y\right) e^{-k\alpha^2 t}$$

at $t=0$, $f(x,y) = x$

$$(x - T_0) = \sum_{n,m} C_{nm} \cos\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{m\pi}{h}y\right)$$

(i) $$\int_0^{\ell} (x-T_0)\cos\left(\frac{n'\pi}{\ell}x\right)dx = \sum_m C_{nm}\frac{\ell}{2}\delta_{nn'}\cos\frac{m\pi}{h}y$$

$$\frac{\ell}{n'\pi}\left(x\sin\frac{n'\pi}{\ell}x\right)\Big|_0^{\ell} - \frac{\ell}{n'\pi}\int_0^{\ell}\sin\left(\frac{n'\pi}{\ell}x\right) - T_0\frac{\ell}{n'\pi}\sin\left(\frac{n'\pi}{\ell}x\right)\Big|_0^{\ell} =$$

$$+\frac{\ell}{n'\pi}\frac{\ell}{n'\pi}\left(\cos n'\pi - 1\right) - \cancelto{0}{\frac{T_0\ell}{n'\pi}\sin(n'\pi)} =$$

$$\left(-\frac{2\ell^2}{n^2\pi^2},\ n \text{ odd}\right) = \frac{\ell}{2}C_{nm}\cos\left(\frac{m\pi}{h}y\right)$$

$$-\left(\frac{2\ell^2}{n^2\pi^2}\right)\int_0^{h}\cos\left(\frac{m\pi}{h}y\right)dy = \frac{\ell}{2}\frac{h}{2}C_{nm} \Rightarrow C_{nm} = 0$$

if $m \neq 0$

if $m=0$ $\rightarrow$ the solution is only the x-part

and

$$C_n\frac{\ell}{2} = -\frac{2\ell^2}{n^2\pi^2} \rightarrow C_n = \frac{-4\ell}{n^2\pi^2}$$

and solution is

$$T(x,y,t) = T_0 - \sum_{\substack{n=\\ n\ \text{odd}}}^{\infty}\frac{4\ell}{n^2\pi^2}\cos\left(\frac{n\pi}{\ell}x\right)e^{-\alpha^2 k^2 t}$$

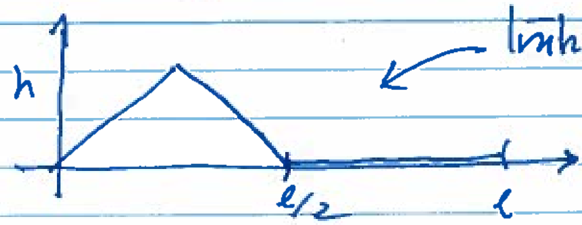
45. B.4.2

a) $\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$, let $\psi(x,t) = f(x)g(t)$

$$\Rightarrow \frac{1}{f}\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2 g}\frac{\partial^2 g}{\partial t^2} = -k^2$$

$$\Rightarrow \psi(x,t) = \left[a_0\cos kx + a_1\sin kx\right]\left[b_0\cos kvt + b_1\sin kvt\right]$$

b)



Initial displacement

for all t, $\psi = 0$ @ $x = 0, l$

$\rightarrow a_0 = 0$, $kl = n\pi \rightarrow k = \frac{n\pi}{l}$

at $t = 0$, $\dot{\psi} = 0$

$$\rightarrow \dot{\psi}(x,t) = a_1 \sin\left(\frac{n\pi x}{l}\right)\left[-b_0 kv\sin kvt + b_1 kv\cos kvt\right]$$

$\rightarrow b_1 = 0$

our solution is thus $\psi(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\overset{\frac{n\pi v}{l}}{kvt}\right)$

c) at $t = 0$, we have the displacement shown in b), find c_n

$$f(x) = \begin{cases} \left(\frac{4h}{l}\right)x, & 0 < x < l/4 \\ \left(2h - \frac{4h}{l}x\right), & l/4 < x < l/2 \\ 0, & x > l/2 \end{cases}$$

at $t=0$

$$f(x)=\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{l}x\right)$$

$$\int_0^l f(x)\sin\left(\frac{n'\pi}{l}x\right)dx = \sum_{n=1}^{\infty} C_n\int_0^l \sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{n'\pi}{l}x\right)dx$$

$$\left.\begin{aligned}&\int_0^{l/4}\frac{4h}{l}x\sin\left(\frac{n'\pi}{l}x\right)dx\\ &+\int_{l/4}^{l/2}\left(2h-\frac{4h}{l}x\right)\sin\left(\frac{n'\pi}{l}x\right)dx\end{aligned}\right\} = C_n\frac{l}{2}\delta_{nn'}.$$

$$\rightarrow C_n\frac{l}{2}\delta_{nn'} = \frac{4h}{l}\left[-x\frac{l}{n'\pi}\cos\left(\frac{n'\pi}{l}x\right)\Big|_0^{l/4} + \frac{l}{n'\pi}\int_0^{l/4}\cos\frac{n\pi}{l}x\,dx\right]$$

$$+2h\left[\frac{-l}{n'\pi}\cos\left(\frac{n'\pi}{l}x\right)\Big|_{l/4}^{l/2}\right] - \frac{4h}{l}\left[-\frac{l}{n'\pi}x\cos\left(\frac{n'\pi}{l}x\right)\Big|_{l/4}^{l/2} + \frac{l}{n'\pi}\int_{l/4}^{l/2}\cos\frac{n\pi}{l}x\,dx\right]$$

$$= +\frac{4h}{l}\left[\frac{l^2}{n\pi 4}\left\{\cos\left(\frac{n'\pi}{4}\right)-1\right\} + \frac{l^2}{n'^2\pi^2}\sin\left(\frac{n'\pi}{l}x\right)\Big|_0^{l/4}\right]$$

$$-\frac{2hl}{n'\pi}\left(\cos\frac{n'\pi}{2}-\cos\frac{n'\pi}{4}\right) - \frac{4h}{l}\left[-\frac{l^2}{2n'\pi}\cos\frac{n\pi}{2} + \frac{l^2}{4n'\pi}\cos\frac{n'\pi}{4} + \frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}-\sin\frac{n\pi}{4}\right)\right]$$

$$C_n = \frac{8h}{l^2}\left[\frac{-l^2}{4n\pi}\left\{\cos\left(\frac{n\pi}{4}\right)-1\right\} + \frac{l^2}{n^2\pi^2}\left\{\sin\frac{n\pi}{4}\right\}\right]$$

$$-\frac{4h}{n\pi}\left(\cos\frac{n\pi}{2}-\cos\frac{n\pi}{4}\right) - \frac{8h}{l^2}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{4n\pi}\cos\frac{n\pi}{4}\right.$$

$$\left.+\frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}-\sin\frac{n\pi}{4}\right)\right]$$

$$\psi(x,t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{2 \sin \frac{n\pi}{4} - \sin \frac{n\pi}{2}}{n^2} \right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi v t}{l}\right)$$

46. The d'Alembert formulation is a neat way to solve the ~~[illegible]~~ 1D wave equation and can be extended to multiple dimensions under certain conditions, however, the notion that a wave propagates w/o changing form seems strict. For example, even a spherical wave changes shape as it propagates, the amplitude of the wave goes down. The d'Alembert formulation was a nice way to teach about PDE characteristics w/o doing the eigenvalue problem.