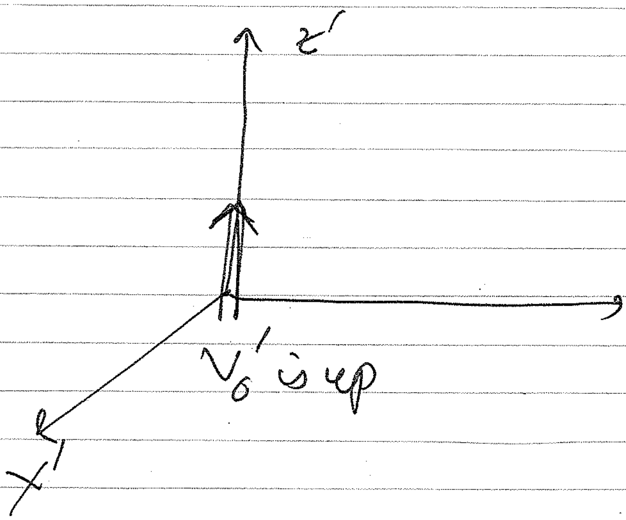
HW #4

5.16

a bullet is fired straight up (z' direction) w/speed v_0'. Assume g is constant & air resistance is negligible. Find where the bullet hits the ground.



a) $$z'(t) = z_0' + \dot{z}_0' t - \frac{1}{2} g t^2 + \omega \dot{x}_0' t^2 \cos\lambda$$

$$x'(t) = x_0' + \dot{x}_0' t - \omega t^2 (\dot{z}_0' \cos\lambda - \dot{y}_0' \sin\lambda) + \frac{1}{3} \omega g t^3 \cos\lambda$$

b) B.C.'s: at $t=0$, $$\boxed{\begin{aligned} &\dot{z}_0' = v_0', \ \dot{x}_0' = \dot{y}_0' = 0 \\ &z_0' = 0 = x_0' = y_0' \end{aligned}}$$

$$\Rightarrow z'(t) = v_0' t - \frac{1}{2} g t^2 \Rightarrow t\left[v_0' - \frac{g}{2} t\right] = z'(t)$$

time of flight is from $z'=0$ to $z'=0$

or $t_f = 2v_0'/g$

c) <u>Deflection</u>: $$x'\left(\frac{2v_0'}{g}\right) = -\omega \overset{v_0'}{\dot{z}_0} \cos\lambda \left[\frac{4v_0'^2}{g^2}\right] + \frac{1}{3} \omega \cancel{g} \cos\lambda \left[\frac{8v_0'^3}{\cancel{g^3}\, g^2}\right]$$

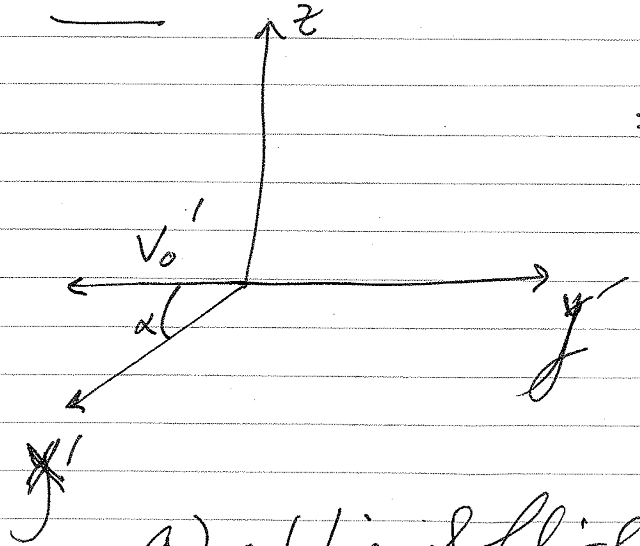
$$= -\frac{4}{3} \omega \cos\lambda \left[\frac{v_0'^3}{g^2}\right]$$

westward

5.17

Suppose we shoot the bullet due east (x' direction) w/ elevation α. Find the deflection (in Northern hemisphere)

Sol^{n}



$$\Rightarrow \boxed{\begin{gathered} x_0' = y_0' = z_0' = 0 \\ \dot{x}_0' = V_0'\cos\alpha,\ \dot{z}_0' = \alpha\sin\alpha, \\ \dot{y}_0' = 0 \end{gathered}}$$

a) get time-of-flight from

$$z'(t) = \cancel{z_0'} + \dot{z}_0' t - \tfrac{1}{2} g t^2 + \omega \dot{x}_0' \cos\lambda\, t^2$$

$$= t\left[\dot{z}_0' - \frac{g}{2} t + \omega \dot{x}_0' \cos\lambda\, t\right]$$

$$\Rightarrow \boxed{t_f = \frac{\dot{z}_0'}{\frac{g}{2} - \omega \dot{x}_0' \cos\lambda}}$$

b) get deflection from

$$y'(t_f) = \cancel{y_0'} + \cancel{\dot{y}_0' t_f} - \omega \dot{x}_0' \sin\lambda\, t_f^2$$

$$= -\omega_0 \dot{x}_0' \sin\lambda \left[\frac{\dot{z}_0'}{\frac{g}{2} - \omega \dot{x}_0' \cos\lambda}\right]^2$$

$$\simeq -\frac{\omega_0 \dot{x}_0' \dot{z}_0'^2 \sin\lambda}{(g^2/4)} \left(1 + \frac{2\omega \dot{x}_0' \cos\lambda}{g}\right)$$

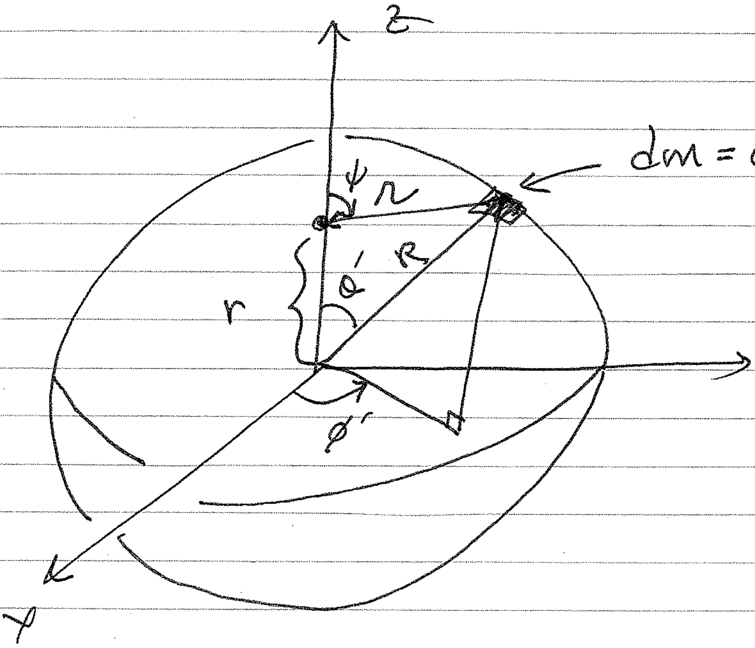
makes the 2^{nd} term in parentheses 2^{nd} order in ω & so, we ignore it

$$\rightarrow \boxed{y'(t_f) \simeq -\frac{4}{g^2}\omega V_0'^3 \cos\alpha \sin^2\alpha \sin\lambda}$$

6.2

Find the field:

a) inside a shell

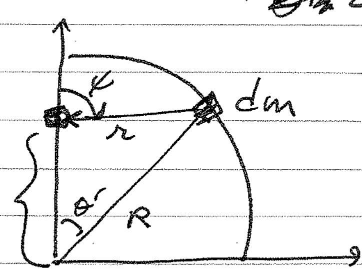


$$dm = \sigma \overbrace{\left[R^2 \sin\theta' \, d\theta' \, d\phi'\right]}^{\text{unit of mass}}$$

① find the field on the z-axis. Because of symmetry, we do not lose anything by doing this

② Note that we need only keep the z-component of $\vec{F}$

$$\Rightarrow \quad dF_z = -G\frac{dm\, M_0}{\mathscr{r}^2}\cos\psi \quad \text{where } \cos\psi = \left(\frac{R^2 - r^2 - \mathscr{r}^2}{2r\mathscr{r}}\right)$$



$$\Rightarrow F_z = -GM_0\sigma\int \frac{R^2 - r^2 - \mathscr{r}^2}{\mathscr{r}^2(2r\mathscr{r})} R^2 \sin\theta' \, d\theta' \, d\phi'$$

$$= -\frac{GM_0R^2}{2r}2\pi\sigma\int\left[\frac{(R^2-r^2)}{\mathscr{r}^3} - \frac{1}{\mathscr{r}}\right]\sin\theta' \, d\theta'$$

$$= -\frac{GM_0\pi R^2\sigma}{r}\int\left[\frac{(R^2-r^2)}{(r^2+R^2-2rR\cos\theta')^{3/2}} - \frac{1}{(r^2+R^2-2rR\cos\theta')^{1/2}}\right]d(-\cos\theta')$$

$$= \frac{GM_0\pi R^2\sigma}{r}\left[\frac{(R^2-r^2)\left(\frac{+\cancel{2}}{\cancel{2}rR}\right)}{(r^2+R^2-2rR\cos\theta')^{1/2}} + \frac{\cancel{2}(r^2+R^2-2rR\cos\theta')^{\frac{1}{2}}}{+\cancel{2}rR}\right]_0^{\pi}$$

$$= \frac{GM_0\pi R^2\sigma}{r}\left[\frac{(R^2-r^2)}{rR}\left\{\frac{1}{\sqrt{(r+R)^2}} - \frac{1}{\underbrace{\sqrt{(r-R)^2}}_{\text{flip order to keep } \sqrt{\ } > 0}}\right\} + \frac{\sqrt{(r+R)^2}-\sqrt{(r-R)^2}}{rR}\right]$$

$$= \frac{GM_0\pi R^2\sigma}{r^2R}\left[\cancel{(R^2-r^2)}\left[\frac{\overbrace{(R-r)-(R+r)}^{-2r}}{\cancel{(R+r)(R-r)}}\right] + \left[\underbrace{(R+r)-(R-r)}_{2r}\right]\right]$$

$= 0$! as desired

b) Find the potential where $\vec{F} = 0$

$$-\int_{\infty}^{r} dV = \int_{\infty}^{r} \vec{F}\cdot d\vec{r}$$

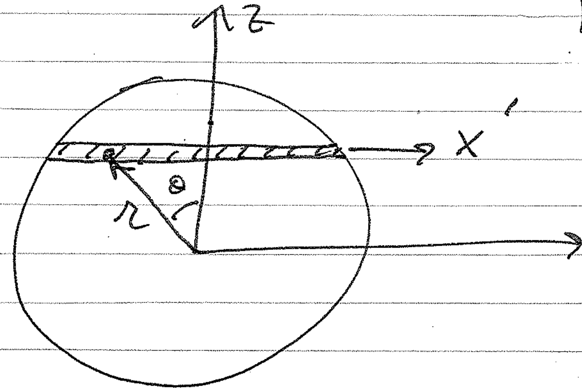
$$= -\int_{\infty}^{R} \frac{G(4\pi R^2\sigma)M_0}{r^2}\,dr + \int_{R}^{r} \cancel{\vec{F}(r<R)\cdot d\vec{r}}^{\,0}$$

$$-V(r) + \cancel{V(\infty)}^{\,0} = +\frac{G(4\pi R^2\sigma)M_0}{R}$$

$$\boxed{V(r) = -\frac{G\,4\pi R^2\sigma M_0}{R} = \text{constant}}$$

6.4 Show that the motion is simple harmonic w/ the same period as 6.3 for a tunnel drilled obliquely through the ⊕ -

Soln



$$\vec{F} = -G\frac{\frac{4\pi}{3}r^3\rho_0 m}{r^2}\hat{r}$$

$$= -\frac{4\pi}{3}G\rho_0 m r\hat{r}$$

(i) we want $\vec{F}$ along x' $\Rightarrow$ $F_{x'} = -\frac{4\pi}{3}G\rho_0 m \underbrace{r\sin\theta}_{x'}$

and the force law is

$$F_{x'} = m\ddot{x}' = -\frac{4\pi}{3}G\rho_0 m x'$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{4\pi}{3}G\rho_0}}$$

6.5 Assume a circular orbit, show that Kepler's 3rd law follows Newton's 2nd law and gravity.

Sol^n

(i) $$\vec{F_r} = (m\ddot{r} - mr\dot{\theta}^2)\hat{r} = -\frac{GMm}{r^2}\hat{r}$$

$$\Rightarrow mR\dot{\theta}^2 = \frac{GMm}{R^2}$$

(ii) $$\vec{F_\theta} = (mr\ddot{\theta} + 2\dot{r}\dot{\theta}m) = 0 \Rightarrow mR\dot{\theta} = \text{constant}$$

(if $\dot{r} = 0$)

and so, $$mR\left(\frac{2\pi}{P}\right)^2 = \frac{GMm}{R^2}$$

$$\Rightarrow \boxed{P^2 = \frac{4\pi^2}{GM}R^3}$$

6.11

A particle moving in inverse-cube field of force. Show that, in addition to the exponential spiral orbit of 6.10 two other types of orbit possible & give their equations

Soln

(i) $\vec{f}(r) = -\frac{k}{r^3}\hat{r} = -k u^3 \hat{r}$, $k > 0$

(ii) $\frac{d^2u}{d\theta^2} + u = -\frac{f(u)}{ml^2u^2} = \frac{k}{ml^2} u$

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{k}{ml^2}\right) u = 0$$

(iii) Solutions:

a) $(1 - k/ml^2) = 0 \rightarrow u = u_0 + u_1\theta$

$$\boxed{r = \frac{1}{u_0 + u_1\theta}}$$

if $u_1 = 0 \Rightarrow r =$ constant, circular orbit

if $u_1 \neq 0 \Rightarrow$ decaying spiral

b) $u = A\cos\left(\sqrt{1 - \frac{k}{ml^2}}\,\theta - \theta_0\right)$

$$\Rightarrow r = \frac{1}{A\cos\left(\sqrt{1 - \frac{k}{ml^2}}\,\theta - \theta_0\right)}$$

set $\theta_0 = 0$

$$\boxed{r(\theta) = \frac{1}{A\cos\sqrt{1 - \frac{k}{ml^2}}\,\theta}}$$

(1) $1 - \frac{k}{m\ell^2} < 0 \Rightarrow \sqrt{1 - \frac{k}{m\ell^2}} = i\sqrt{\frac{k}{m\ell^2} - 1}$

ad $r(\theta) = \dfrac{1}{A\left\{\dfrac{e^{-\theta\sqrt{k/m\ell^2 - 1}} + e^{+\theta\sqrt{k/m\ell^2-1}}}{2}\right\}}$

$$\boxed{r(\theta) = \frac{1}{A\cosh\sqrt{\frac{k}{m\ell^2} - 1}\,\theta}} \leftarrow \text{"spiraly"}$$

(2) $1 - \frac{k}{m\ell^2} > 0 \Rightarrow r(\theta) = \dfrac{1}{A\cos\sqrt{1 - \frac{k}{m\ell^2}}\,\theta}$

if $\sqrt{1 - \frac{k}{m\ell^2}}\,\theta = \pm\frac{\pi}{2} \rightarrow r(\theta) \longrightarrow \infty$

ad not periodic

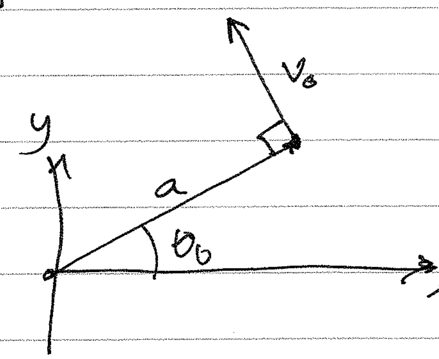
6.14

A particle of mass is projected w/ velocity V_0 at right angles to the radius vector at a distance a from the origin of a center of an attractive force,

$$\vec{f}(r) = -k\left[\frac{4}{r^3}+\frac{a^2}{r^5}\right]\hat{r}$$

$\boxed{\text{If } mV_0^2 = 9k/2a^2,}$

(a) find $r(\theta)$. Use the energy equation,



$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r),$$

where $V(r) = -k\left[\frac{2}{r^2}+\frac{a^2}{4r^4}\right]$, from $f(r)$ above

(i) at $t=0$, $\dot{r}=0$ & $r\dot{\theta} = V_0$ note: $mV_0^2 = \frac{9k}{2a^2}$

$$\Rightarrow E = \frac{m}{2}V_0^2 - k\left[\frac{2}{a^2}+\frac{1}{4a^2}\right] = 0$$

also, $\boxed{E = 0 = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)}$

(ii) recall: $\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{dr}{d\theta} = \frac{\ell}{r^2}\frac{dr}{d\theta} = +\ell u^2\left(-\frac{1}{u^2}\right)\frac{du}{d\theta}$ $u=\frac{1}{r} \rightarrow r=\frac{1}{u}$

$$\Rightarrow 0 = \cancel{\frac{m}{2}}\left(\left[\cancel{\ell}\frac{du}{d\theta}\right]^2 + \frac{\cancel{\ell^2}}{r^2}\right) - \frac{4\cancel{2}k}{m\ell^2}\left(u^2+\frac{a^2}{8}u^4\right)$$

$$\Rightarrow \int \frac{du}{\sqrt{u^2\left(\frac{4k}{m\ell^2}-1\right)+\frac{ka^2}{2m\ell^2}u^4}} = \int d\theta$$

switch back to r

$$\int \frac{-\frac{1}{r^2}dr}{\sqrt{\frac{ka^2}{2m\ell^2}\frac{1}{r^4}+\left(\frac{4k}{m\ell^2}-1\right)\frac{1}{r^2}}} = \int d\theta$$

$$= -\frac{1}{\sqrt{\left(\frac{4k}{ml^2}-1\right)}} \int \frac{dr}{\sqrt{r^2 + \frac{ka^2}{8k-2ml^2}}} = \theta - \theta_0$$

note: at $t=0$, $l = av_0$ & $mv_0^2 = \frac{9k}{2a^2}$ $\Longrightarrow \frac{ka^2}{8k-2ml^2} = \frac{ka^2}{8k-2\cancel{m}\left(\frac{9k}{2\cancel{m}}\right)}$

$$= -a^2$$

$\Rightarrow l^2 = \frac{9k}{2\cancel{a^2}}\frac{1}{m}\cancel{a^2}$ and also

$$\frac{4k}{ml^2} = \frac{4\cancel{k}}{\cancel{m}}\frac{2\cancel{m}}{9\cancel{k}} = \frac{8}{9}$$

$$= -\sqrt{-9}\int \frac{dr}{\sqrt{r^2-a^2}} = \theta - \theta_0$$

$$\Longrightarrow -3\cancel{i}\int \frac{dr}{\cancel{i}\sqrt{(a^2-r^2)}} = \theta - \theta_0$$

$$3\cos^{-1}\left(\frac{r}{a}\right)\Big|_{r=a}^{r} = \theta - \theta_0$$

$$\cos^{-1}\left(\frac{r}{a}\right) = \frac{\theta-\theta_0}{3}$$

$$\boxed{r = a\cos\left(\frac{\theta-\theta_0}{3}\right)}$$

(b) How long does it take the particle to travel through an angle $3\pi/2$? Where is the particle at this time?

(i) $r(\theta) = a\cos\left(\frac{\theta-\theta_0}{3}\right)$ set $\theta_0 = 0$

(ii) $l = r^2\dot{\theta}$ $\longrightarrow$ $\dot{\theta} = \frac{l}{r^2} = \frac{l}{a^2\cos^2(\theta/3)}$

$$\int \cos^2\frac{\theta}{3}\, d\theta = \frac{l}{a^2}\int dt$$

$$3\left[\frac{\theta}{6} + \frac{1}{4}\sin\frac{2\theta}{3}\right] = \frac{l}{a^2}t$$

$$\rightarrow t = \frac{3a^2}{6\ell}\left(\frac{3\pi}{2} - 0 + \frac{3}{2}\left[\cancel{\sin\pi} - \sin 0\right]\right)$$

$$\boxed{t = \frac{3\pi}{4}\left(\frac{a^2}{\ell}\right)}$$

(c) $\boxed{r\left(\frac{3\pi}{2}\right) = a\cos\left(\frac{3\pi/2}{3}\right) = 0}$

(d) what is v at this time?

(i) $r(\theta) = a\cos\left(\frac{\theta}{3}\right) \rightarrow r\left(\theta = \frac{3\pi}{2}\right) = 0$

(ii) $E = 0 = \frac{m}{2}v^2 - \frac{2k}{r^2} - \frac{ka^2}{4r^4} \rightarrow v^2 = \frac{4k}{mr^2} + \frac{ka^2}{2mr^4}$

$\rightarrow \infty$