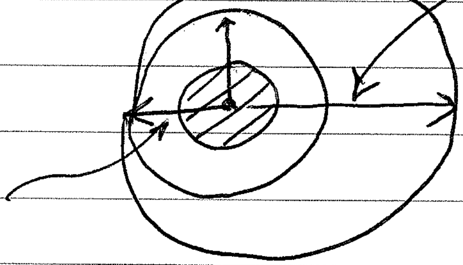
HW #5

6.15

a) Find the fractional change in the apogee $(\delta r_0 / r_1)$ as a function of a small boost to circular orbit speed.



(i) the new orbit has eccentricity, $\epsilon = \left(\dfrac{\overbrace{r_{ap} - r_1}^{\delta r_1}}{r_{ap} + r_1} \right)$

where $r_0 = r_1$ = orig'nl radius

(ii) the eccentricity of new orbit is

$$\epsilon = \sqrt{1 + \frac{2E}{k}\frac{m l^2}{k}}$$

(iii)

$$E = \frac{1}{2} m \left(V_c + \delta V\right)^2 - \frac{G M_\oplus m}{r_\oplus} \quad \text{after boost by } \delta v$$

$$l = (V_c + \delta V) r_1 \quad \text{after boost by } \delta v$$

(iv)

$$\epsilon^2 = 1 + \frac{2m}{k^2}\left[\overbrace{\left(\frac{m}{2}\left\{V_c^2 + 2V_c \delta V + \delta V^2\right\} - \frac{G M_\oplus m}{r_1}\right)}^{E} \overbrace{r_1^2 \left(V_c^2 + 2V_c \delta V + \delta V^2\right)}^{l^2}\right]$$

$$= 1 + \frac{2m}{k^2}\left[\left(\frac{m}{2}V_c^2 - \frac{G M_\oplus m}{r_1}\right) V_c^2 r_1^2 + \left(\frac{m}{2}V_c^2 - \frac{G M_\oplus m}{r_1}\right) r_1^2 \left(2V_c \delta V + \delta V^2\right)\right.$$

$$+ \left(m V_c \delta V + \frac{m}{2}\delta V^2\right) r_1^2 V_c^2$$

$$\left. + \left(m V_c \delta V + \frac{m}{2}\delta V^2\right) r_1^2 \left(2V_c \delta V + \delta V^2\right)\right]$$

note: ① $\dfrac{GM_{\oplus}m}{r_1^2} - \dfrac{mV_c^2}{r_1} = 0 \Rightarrow \boxed{\dfrac{GM_{\oplus}\cancel{m}}{r_1} = \cancel{m}V_c^2}$

② $1 + \dfrac{2m}{k^2}\left(\dfrac{m}{2}v_c^2 - \dfrac{GM_{\oplus}m}{r_1}\right)v_c^2 r_1^2 = \epsilon^2_{\text{original}} = 0$

$$\Rightarrow \epsilon^2 = \frac{2m}{k^2}\Big\{ -\frac{m}{2}v_c^2 r_1^2\left(\cancel{2v_c\delta v} + \delta v^2\right) + \left(\cancel{m v_c\delta v} + \frac{m}{2}\delta v^2\right) r_1^2 v_c^2 + \left(m v_c\delta v + \frac{m}{2}\delta v^2\right) r_1^2\left(2v_c\delta v + \delta v^2\right)\Big\}$$

$$= \frac{2m}{k^2}\Big\{ m r_1^2\left[2V_c^2\delta V^2 + V_c\delta V^3 + V_c\delta V^3 + \frac{1}{2}\delta V^4\right]\Big\}$$

$$\simeq \frac{4m^2v_c^2}{k^2}r_1^2\,\delta V^2\left(1 + \frac{\delta V}{2V_c}\right)^2$$

$$\epsilon^2 = 4\frac{\delta V^2}{V_c^2}\left(1 + \frac{\delta V}{2V_c}\right)^2$$

(v) $\epsilon = 2\dfrac{\delta V}{V_c} = \dfrac{\delta r}{r_{apo} + r_1} = \dfrac{\delta r}{r_1}\left(\dfrac{1-\epsilon}{2}\right)$

$$\Rightarrow \frac{\delta r}{r_1} = 2\frac{\epsilon}{1-\epsilon} \simeq 2\epsilon(1+\epsilon)$$

$$\boxed{\frac{\delta r}{r_1} \simeq 4\frac{\delta v}{v_c}}$$

for small $(\delta v/v_c)$

formula given in the text

(vi) for arbitray δV,

$$\epsilon = 2\left(\frac{\delta V}{V_c}\right) + \left(\frac{\delta V}{V_c}\right)^2 \Rightarrow \boxed{\left(\frac{\delta V}{V_c}\right) = \sqrt{1+\epsilon} - 1}$$

b) if the speed ratio is 1% too big, how much does the rocket miss the Moon?

(i) $\frac{V_0}{V_c} = \frac{V_c + \delta V}{V_c} \Rightarrow$ rocket goes into orbit $\Rightarrow$

$$\epsilon = \frac{60R_\oplus - R_\oplus}{61R_\oplus} = \frac{59}{61}$$

this assumption is not clear ?

(ii) Suppose $\frac{V_0 + 0.01V_c}{V_c}$ is applied

$$\Rightarrow \frac{V_c + \delta V + 0.01V_c}{V_c} = \frac{V_c + \delta V}{V_c} + 0.01 = \sqrt{1+\epsilon} - 1$$

$$1.41 = \sqrt{1+\epsilon}$$

$$\Rightarrow \epsilon = 0.9881$$

vs

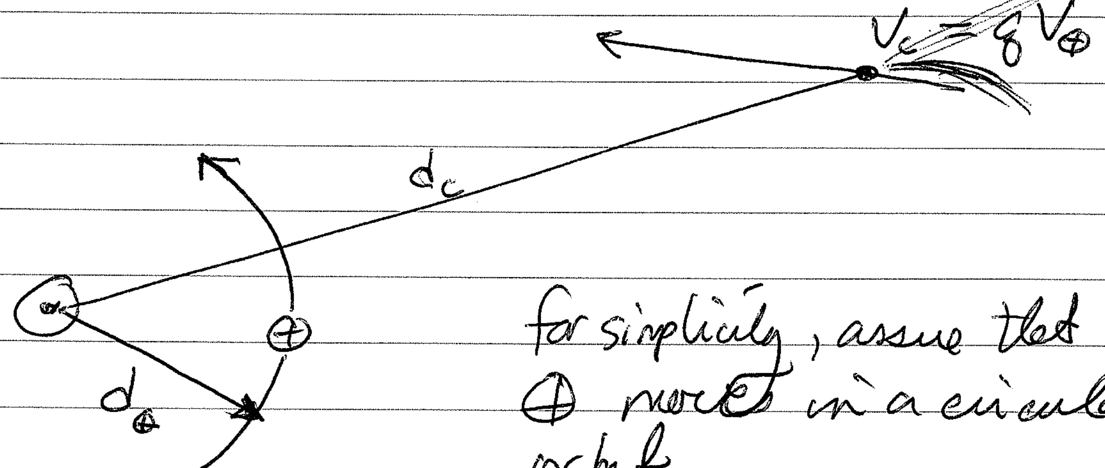
original $\epsilon = 0.96$

$\epsilon = 0.988 \Rightarrow r_{apogee} \approx 166\ R_\oplus$ as opposed to $60R_\oplus$!

6.17

A comet is first seen at a distance d_c A.U. from the $\odot$ and it is traveling w/ speed q times the $\oplus$'s speed. Show that the orbit is hyperbolic (~~i.e. $E>0$~~), parabolic, or elliptic depending on whether the quantity $q^2 d_c \gtreqless 2$.

Sol^{n}



for simplicity, assume that the $\oplus$ moves in a circular orbit

(i) $$E = \frac{m}{2}(qV_\oplus)^2 - \frac{GM_\odot m}{d_c}$$

Compare E to 0 (find sign of E)

$$\Rightarrow E = 0 \text{ when } q^2 d_c = \frac{2GM_\odot \cancel{m}}{\cancel{m}V_\oplus^2}$$

$$= 2\left(\cancel{\frac{GM_\odot}{d_\oplus V_\oplus^2}}\right) d_\oplus \quad 1$$

$$\Rightarrow \boxed{q^2\left(\frac{d_c}{d_\oplus}\right) = 2 \text{ for } E=0}$$

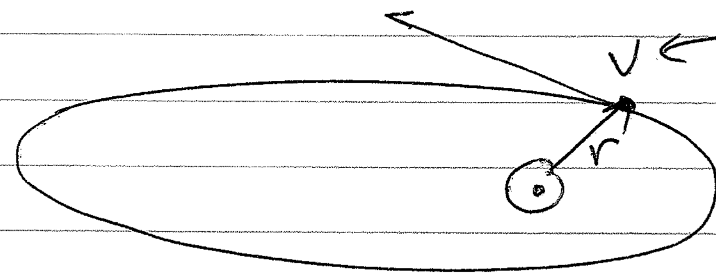
> 2 for $E>0$

< 2 for $E<0$

6.19

at a certain point in its elliptical orbit about the $\odot$, a planet receives a small tangential impulse so that its velocity changes from v to $(v+\delta v)$. Find the resultant change in a, the semi-major axis.

Sol^{n}



"kick" the planet here so that
$v \longrightarrow v+\delta v$

$$\text{(i)}\quad \underbrace{E = -\frac{k}{2a} = -\frac{GM_\odot m}{2a} = \frac{mv^2}{2} - \frac{GM_\odot m}{r}}_{\text{original } E \text{ and } a}$$

$$\text{(ii)}\quad \underbrace{E' = -\frac{k}{2a'} = -\frac{GM_\odot m}{2a'} = \frac{m}{2}(v+\delta v)^2 - \frac{GM_\odot m}{r}}_{\text{perturbed } E \text{ \& } a}$$

$$E+\delta E = E' = -\frac{GM_\odot m}{2(a+\delta a)} = \frac{m}{2}(v+\delta v)^2 - \frac{GM_\odot m}{r}$$

$$-\frac{GM_\odot m}{2a}\left(1-\frac{\delta a}{a}\right) \simeq \frac{m}{2}v^2\left(1+2\frac{\delta v}{v}\right) - \frac{GM_\odot m}{r}$$

$$\Rightarrow \frac{GM_\odot \cancel{m}}{2a}\left(\frac{\delta a}{a}\right) \simeq \cancel{2}\frac{\cancel{m}}{\cancel{2}} v\,\delta v$$

$$\left(\frac{\delta a}{a}\right) \simeq \left[\frac{2av^2}{GM_\odot}\right]\left(\frac{\delta v}{v}\right)$$

original a and v

$$\Longrightarrow \left[\frac{2av^2}{GM_\odot}\right] = \left[\frac{2a\,mv^2}{GM_\odot m}\right] = \left[\frac{mv^2}{E}\right]$$

original E

6.24

Show that that the stability condition for a circular orbit of radius a is equivalent to the condition that

$$\frac{d^2 U}{dr^2} > 0$$

for $r = a$.

Sol^n

(a) (i) Stability Condition is

$$f(a) + \frac{a}{3} f'(a) < 0$$

(ii) $$U(r) = \frac{ml^2}{2r^2} + \cancel{V}(r)$$

(iii) for a conservative force $\vec{f}(r) = -\vec{\nabla} V(r)$

for a central force $\Longrightarrow$ $f(r) = -\frac{\partial V}{\partial r}$

$$f'(r) = -\frac{\partial^2 V}{\partial r^2}$$

(iv) $$\frac{\partial U}{\partial r} = -\frac{ml^2}{r^3} + \frac{\partial V}{\partial r}, \quad \frac{\partial^2 U}{\partial r^2} = +\frac{3ml^2}{r^4} + \frac{\partial^2 V}{\partial r^2}$$

$$\Rightarrow f(a) + \frac{a}{3} f'(a) < 0$$

$$= -\left(\frac{\partial U}{\partial r} + \frac{m l^2}{a^3}\right) - \frac{a}{3}\left(\frac{\partial^2 U}{\partial r^2} - 3\frac{m l^2}{a^4}\right)$$

0 for circular orbit

$$= -\cancel{\frac{\partial U}{\partial r}} - \frac{a}{3}\frac{\partial^2 U}{\partial r^2}$$

$$= -\frac{a}{3}\frac{\partial^2 U}{\partial r^2}$$

$a > 0$, by definition and $f(a) + \frac{a}{3} f'(a) < 0$

$$\Rightarrow \boxed{\frac{\partial^2 U}{\partial r^2} > 0}$$

(b) Can also argue graphically (as in class)

6.29

for $V(r) = -\frac{k}{r}\left(1+\frac{\epsilon}{r^2}\right)$ and $\epsilon = \frac{2}{5}(R\Delta R)$

find the apsidal angle for a satellite moving in a nearly circular orbit in the equatorial plane of the $\oplus$.

Sol^n

(6.13.3) $\psi = \pi\left[3 + a\frac{f'(a)}{f(a)}\right]^{-\frac{1}{2}}$

(i) $f(r) = -\frac{k}{r^2} - \frac{3\epsilon k}{r^4}$

$f'(r) = \frac{2k}{r^3} + \frac{12\epsilon k}{r^5}$

(ii) $-\frac{k}{r^2} - \frac{3\epsilon k}{r^4} + \frac{mv^2}{r} = -\frac{k}{r^2} - \frac{3\epsilon k}{r^4} + \frac{ml^2}{r^3} = 0$

$\Rightarrow -kr^2 + ml^2 r - 3\epsilon k = 0$ for circular orbit

$\Rightarrow a = \frac{-ml^2 \pm \sqrt{m^2l^2 - 12k^2\epsilon}}{-2k}$

$= \frac{ml^2}{2k}\left(1 \mp \sqrt{1 - 12\epsilon\left(\frac{k}{ml}\right)^2}\right)$

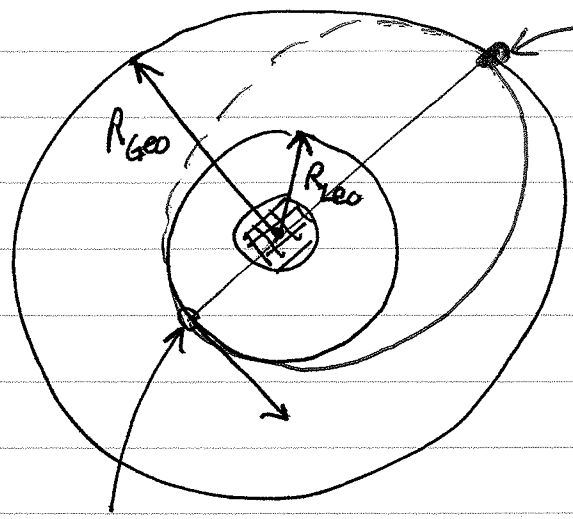
(iii) $$\psi = \pi \left[3 + \cancel{a} \frac{\left(\frac{2\cancel{k}}{\cancel{a}^3} + \frac{12\epsilon \cancel{k}}{\cancel{a^5} a^2} \right)}{-\left(\frac{\cancel{k}}{\cancel{a^2}} + \frac{3\epsilon \cancel{k}}{a^2 \cancel{a^4}} \right)} \right]^{-\frac{1}{2}}$$

$$= \pi \left[3 - \left(\frac{2 + 12\epsilon/a^2}{1 + 3\epsilon/a^2} \right) \right]^{-1/2}$$

$$= \pi \left[3 - 2 \left(\frac{1 + \frac{\cancel{24}\,12}{5} \frac{R\Delta R}{a^2}}{1 + \frac{6}{5} \frac{R\Delta R}{a^2}} \right) \right]^{-\frac{1}{2}}$$

$$\boxed{\psi = \pi \left[\frac{1 - \frac{6}{5} \frac{R\Delta R}{a^2}}{1 + \frac{6}{5} \frac{R\Delta R}{a^2}} \right]^{-\frac{1}{2}}}$$

6.32



rendezous at apogee of transfer orbit

$$\Rightarrow \epsilon = \frac{R_{Geo} - R_{Leo}}{R_{Geo} + R_{Leo}}$$

apply impulse here

a) What is the period for the transfer orbit?

(i) Use Kepler's 3rd law, $P^2 \propto a^3$ or $P^2 = c_0 a^3$ constant

① $P_{Leo}^2 = c_0 a_{Leo}^3$, $P_{Transfer}^2 = c_0 a_{Transfer}^3$

$$\rightarrow \left(\frac{P_{Transfer}}{P_{Leo}}\right)^2 = \left(\frac{a_{Transfer}}{a_{Leo}}\right)^3 \rightarrow \boxed{P_{Transfer} = P_{Leo}\left(\frac{a_{Transfer}}{a_{Leo}}\right)^{3/2}}$$

② $P_{Transfer} = 1.51^h \left(\dfrac{\frac{1}{2}\left[R_{Leo} + R_{Geo}\right]}{R_{Leo}}\right)^{3/2}$

$= 12.8^h$

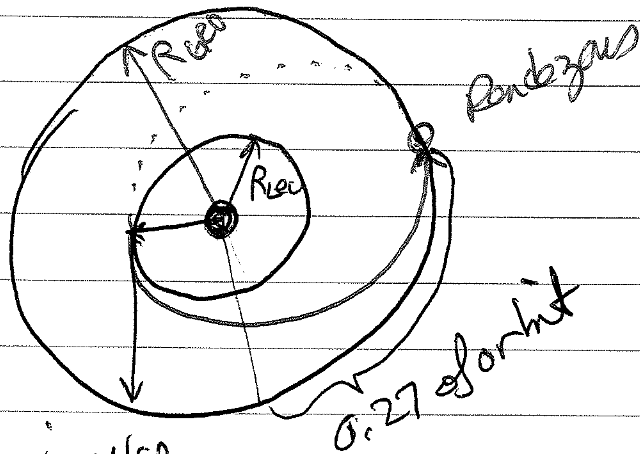
$$\Rightarrow \boxed{\text{trip takes } \tfrac{1}{2} P_{Transfer} = 6.4^h}$$

(ii) for $\frac{1}{2}$ of an orbit Trip takes = 6.4 h

b) where must B be in order to rendezvous in 6.4 h?

$$\text{fraction of orbit} = \frac{6.4 \text{ hours}}{24 \text{ hours}} = 0.27$$

or



6.33

(a) for $f(r) = \frac{k}{r^3}$

$$\rightarrow u'' + u = -\frac{f}{ml^2u^2} = -\frac{k}{ml^2}u$$

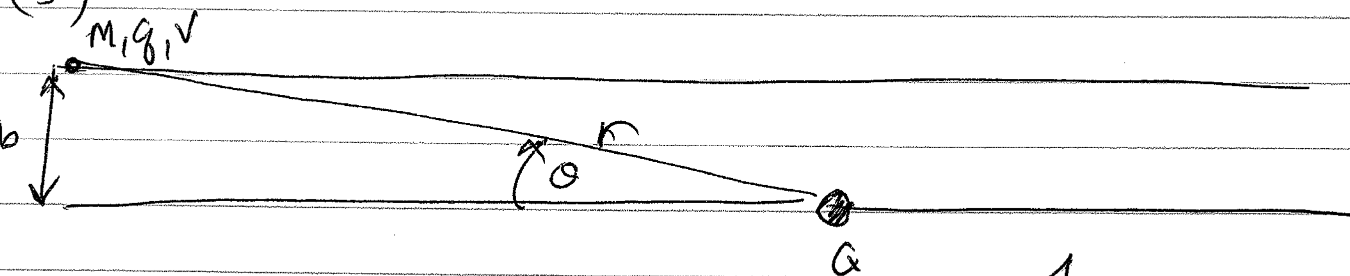
$$\Rightarrow u'' + u\left(1 + \frac{k}{ml^2}\right) = 0$$

or $$\boxed{u = A\cos\sqrt{1+\frac{k}{ml^2}}\,(\theta - \theta_0)}$$

or

$$r = \frac{1}{A\cos\sqrt{1+\frac{k}{ml^2}}\,(\theta-\theta_0)}$$

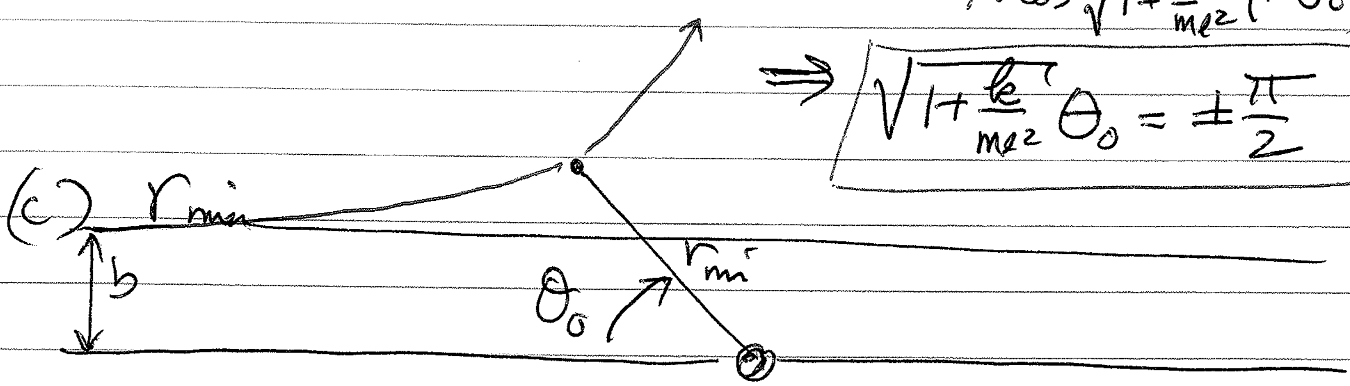
(b)



at $r \to \infty$, let $\theta \to 0 \Rightarrow \infty = \frac{1}{A\cos\sqrt{1+\frac{k}{ml^2}}\,(-\theta_0)}$

$$\Rightarrow \boxed{\sqrt{1+\frac{k}{ml^2}}\,\theta_0 = \pm\frac{\pi}{2}}$$

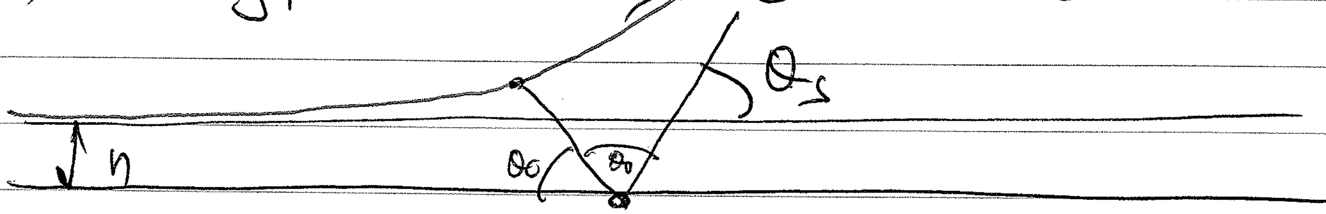
(c)



r_{min} occurs when $\cos \to 1$ or when $\sqrt{1+\frac{k}{ml^2}}\,(\theta-\theta_0) = 0$

$$\Rightarrow \theta = \theta_0$$

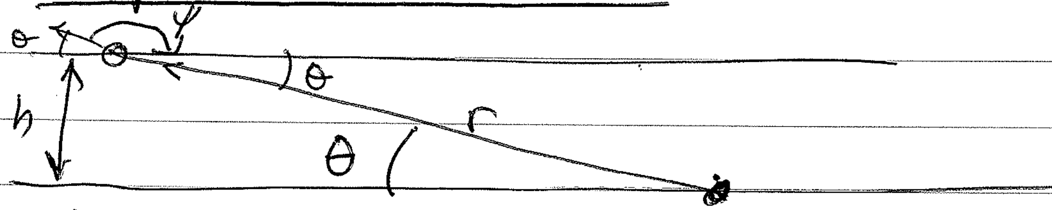
(d) runs off to $r = \infty \Rightarrow \theta \to 2\theta_0$



$\Rightarrow$ Scattering $\measuredangle$, $\boxed{\theta_s = -2\theta_0 + \pi}$

(e) from $a, b, c \Rightarrow$ $\boxed{\theta_s = \pi - \dfrac{\pi}{2\sqrt{1+\frac{k}{me^2}}}}$

(f) Impact Parameter & θ_s



$$\vec{l} = \vec{r} \times \vec{v} \Rightarrow |\vec{l}| = |r v_0 \sin\psi| = |r v_0 \sin(\pi - \theta)|$$

$$l = r v_0 \underbrace{\sin\theta}_{\sin\theta = \frac{b}{r}}$$

$$\boxed{l = v_0 b}$$

$$\Rightarrow \sqrt{1 + \frac{k}{me^2}} = \left(\frac{-\pi}{\theta_s - \pi}\right)$$

$$\frac{\pi^2}{(\theta_s - \pi)^2} = 1 + \frac{k}{m v_0^2 b^2} \Rightarrow \boxed{b^2 = \frac{\frac{k}{m v_0^2}}{\frac{\pi^2}{(\theta_s - \pi)^2} - 1}}$$

(g) find $\sigma(\theta_s)d\Omega = 2\pi|b\,db|$. okay we have from (f)

$$\boxed{b^2 = \frac{k}{mv_0^2}\left[\frac{(\theta_s-\pi)^2}{\pi^2-(\theta_s-\pi)^2}\right]}$$

$$\cancel{2}b\,db = \frac{k}{mv_0^2}\left[\frac{\cancel{2}(\theta_s-\pi)}{\pi^2-(\theta_s-\pi)^2} - \frac{(-\cancel{2})(\theta_s-\pi)^3}{\{\pi^2-(\theta_s-\pi)^2\}^2}\right]d\theta_s$$

$$b\,db = \frac{k}{mv_0^2}\left[\frac{\theta_s-\pi}{\pi^2-(\theta_s-\pi)^2}\right]\left[1+\frac{(\theta_s-\pi)^2}{\pi^2-(\theta_s-\pi)^2}\right]d\theta_s$$

$$= \frac{k}{mv_0^2}\left(\frac{\theta_s-\pi}{\pi^2-(\theta_s-\pi)^2}\right)\left[\frac{\pi^2}{\pi^2-(\theta_s-\pi)^2}\right]d\theta_s$$

$$= \frac{\pi^2 k}{mv_0^2}\left[\frac{(\theta_s-\pi)}{\{\pi^2-(\theta_s-\pi)^2\}^2}\right]d\theta_s$$

$E = \frac{mv_0^2}{2}$

$$= \frac{\pi^2 k}{2E}\left[\frac{(\theta_s-\pi)}{(\cancel{\pi^2}-\theta_s^2+2\pi\theta_s-\cancel{\pi^2})^2}\right]d\theta_s$$

$$= \frac{\pi^2 k}{2E}\left[\frac{(\theta_s-\pi)}{\theta_s^2(2\pi-\theta_s)^2}\right]d\theta_s$$

$$\sigma(\theta_s)d\Omega = 2\pi|b\,db|\ \cancel{d\theta_s}$$

$$\boxed{\sigma(\theta_s)d\Omega = \frac{\pi^3 k}{E}\left[\frac{(\pi-\theta_s)}{\theta_s^2(2\pi-\theta_s)^2}\right]d\theta_s}$$