HW #1

4.2

By finding the curl, determine which of the following are conservative:

a) $\vec{F} = \hat{i}x + \hat{j}y + \hat{k}z$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left(\cancelto{0}{\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y},\ \cancelto{0}{\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z},\ \cancelto{0}{\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x}\right)$$

conservative

$= 0$ ✓

b) $\vec{F} = \hat{i}y - \hat{j}x + \hat{k}z^2$

$$\vec{\nabla}\times\vec{F} = \left(\cancelto{0}{\frac{\partial}{\partial y}z^2 - \frac{\partial}{\partial z}y},\ \cancelto{0}{\frac{\partial}{\partial z}y - \frac{\partial}{\partial x}z},\ \cancelto{-2}{-\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}y}\right)$$

not conservative

$\neq 0$ ✓

c) $\vec{F} = \hat{i}y + \hat{j}x + \hat{k}z^3$

$$\vec{\nabla}\times\vec{F} = \left(\cancelto{0}{\frac{\partial}{\partial y}z^3 - \frac{\partial}{\partial z}x},\ \cancelto{0}{\frac{\partial}{\partial z}y - \frac{\partial}{\partial x}z^3},\ \cancelto{0}{\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}y}\right)$$

$= 0$ ✓

conservative

d) $\vec{F} = -kr^{-n}\hat{e}_r \rightarrow F_\theta = F_\phi = 0$

$$\vec{\nabla}\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta\,F_\phi \end{vmatrix} = \frac{\left(\cancelto{0}{-\frac{\partial}{\partial\phi}rF_\theta},\ \cancelto{0}{r\frac{\partial}{\partial\phi}F_r},\ \cancelto{0}{-\frac{\partial}{\partial\theta}F_r}\right)}{r^2\sin\theta}$$

conservative

$= 0$

4.3

Find the value of c $\ni$ each of the following forces is conservative:

a) $\vec{F} = \hat{i}xy + \hat{j}cx^2 + \hat{k}z^3$

$\vec{\nabla} \times \vec{F} = 0 \rightarrow$ (i) $\frac{\partial}{\partial y}(z^3) - \frac{\partial}{\partial z}(cx^2) = 0 \rightarrow$ nothing about c

(ii) $\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(z^3) = 0 \rightarrow$ nothing about c

(iii) $\frac{\partial}{\partial x} cx^2 - \frac{\partial}{\partial y}(xy) = 2cx - x = 0$

$\Longrightarrow \boxed{c = 1/2}$

b) $\vec{F} = \hat{i}\left(\frac{z}{y}\right) + \hat{j}c\left(\frac{xz}{y^2}\right) + \hat{k}\left(\frac{x}{y}\right)$

$$\vec{F} = -\vec{\nabla} V \Longrightarrow \begin{cases} V = x\frac{z}{y} + f(y,z), \\ V = c\left(-\frac{xz}{y}\right) + g(x,z), \\ V = \left(\frac{x}{y}\right)z + h(x,y) \end{cases}$$

if $f = g = h = 0 \Longrightarrow \boxed{c = -1}$

4.4

A particle of mass m moving in 3 D under the potential energy function

$$V(x,y,z) = \alpha x + \beta y^2 + \gamma z^2$$

has speed v_0 when it passes through the origin $(0,0,0)$.

a) What is its speed when it passes through $(1,1,1)$?

Use $E = \frac{1}{2}mv^2 + V(x,y,z)$

at $(x,y,z) = (0,0,0)$, $v = v_0$

$$\Longrightarrow E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + V(x,y,z)$$

and so,

$$\boxed{v = \sqrt{\frac{2}{m}}\sqrt{\frac{m}{2}v_0^2 - (\alpha + \beta + \gamma)}}$$

at $(x,y,z) = (1,1,1)$

b) If $(1,1,1)$ is a turning point in the motion $(v=0)$, what is v_0?

$$v = 0 = \sqrt{\frac{2}{m}}\sqrt{\frac{m}{2}v_0^2 - (\alpha+\beta+\gamma)}$$

$$\longrightarrow \boxed{v_0 = \sqrt{\frac{2}{m}}\sqrt{\alpha + \beta + \gamma}}$$

c) What is the Equation of Motion?

$$\vec{F} = m\frac{d}{dt}\vec{v} = -\vec{\nabla} V$$

$$= -\hat{i}(\alpha) - \hat{j}(2\beta y) - \hat{k}(2\gamma z)$$

$$\Rightarrow \boxed{\begin{aligned} m\frac{d^2x}{dt^2} &= -\alpha \\ m\frac{d^2y}{dt^2} &= -2\beta y \\ m\frac{d^2z}{dt^2} &= -2\gamma z \end{aligned}}$$

4.5

Consider the two force functions:

a) $\vec{F} = \hat{i}x + \hat{j}y$

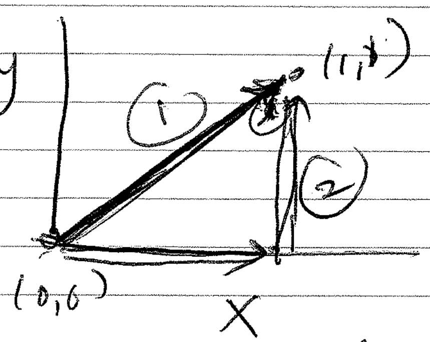
b) $\vec{F} = \hat{i}y - \hat{j}x$

Verify that (a) is conservative and (b) is nonconservative by showing that

$$\int \vec{F} \cdot d\vec{r}$$

is independent of path for (a) but not for (b). Consider the interval $(0,0) \to (1,1)$. For one path use $x = y$ and for the other path that $(0,0) \to (1,0)$ and then the line $x=1$ to $(1,1)$.

Paths



Path 1: $d\vec{r} = (\hat{i}dx + \hat{j}dy)$

$= dx(\hat{i} + \frac{dy}{dx}\hat{j})$

$= dx(\hat{i} + \hat{j})$

path 1

$x = y \to dx = dy$

a) $\int \vec{F} \cdot d\vec{x} = \int (x dx + y dy) = \int (x dx + x dx)$

$= 2\int x dx$

$= 1$

path 2

$$= \int_{0,\, y=0}^{1} x dx + \int_{0,\, x=1}^{1} y dy = \frac{1}{2} + \frac{1}{2} = 1$$

path 1

b) $\int \vec{F} \cdot d\vec{x} = \int (y dx - x dy) = \int (x dx - x dx) = 0$

path 2

$$= \int_0^1 \overset{y=0}{y} dx - \int_0^1 \overset{x=1}{x} dy = 0 - 1$$

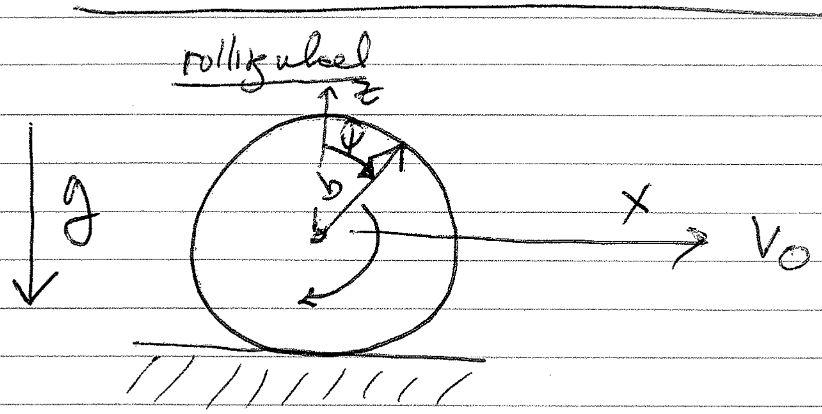
4.7

Particles of mud are thrown from the rim of a rolling wheel. If the forward speed of the wheel is V_0 and the radius of the wheel is b, show that the greatest height above the ground that the mud can go is

$$h_{max} = b + \frac{V_0^2}{2g} + \frac{gb^2}{2V_0^2}$$

at what point on the rolling wheel does this mud leave? Assume that

$$V_0^2 \geq bg$$



a point on the rim of the wheel has velocity:

$$① \quad \dot{x} = V_0 + b\left(\frac{V_0}{b}\right)\cos\psi$$

$$② \quad \dot{z} = b\left(\frac{V_0}{b}\right)\sin\psi$$

(i) Write down the z-equation of motion

$$m\ddot{z} = -mg$$

integrate to find $\dot{z}(t)$ & $z(t)$

$$\Rightarrow \dot{z} = -gt + V_1$$

$$z = -\tfrac{1}{2}gt^2 + V_1 t + h_0$$

(ii) the maximum height is reached when $\dot{z} = 0$

$$\rightarrow \boxed{t_{max} = (V_1/g)}$$

(iii) the maximum height is then

$$z_{max} = -\tfrac{1}{2} g \left(v_1/g \right)^2 + v_1 \left(v_1/g \right) + h_0$$

(iv) what are v_1 & h_0?

$$v_1 = b\left(\frac{v_0}{b}\right)\sin\psi \, , \quad h_0 = b + b\cos\psi$$

$$\Rightarrow z_{max} = -\frac{v_1^2}{2g} + \frac{v_1^2}{g} + h_0$$

$$= \frac{v_0^2}{2g}\sin^2\psi + b(1+\cos\psi)$$

(v) find $\psi \ni z_{max}$ is an extremum (i.e., where $\frac{\partial z_{max}}{\partial \psi} = 0$)

$$\frac{\partial z_{max}}{\partial \psi} = \frac{v_0^2}{g}\sin\psi\cos\psi + b(-\sin\psi) = 0$$

take real $\cos\psi$

$$\Rightarrow \cos\psi = \frac{bg}{v_0^2} \Rightarrow \frac{bg}{v_0^2} \leq 1$$

(as stipulated)

$$\Rightarrow z_{max} = \frac{v_0^2}{2g}\left(1 - \frac{b^2 g^2}{v_0^4}\right) + b\left(1 + \frac{bg}{v_0^2}\right)$$

$$\boxed{z_{max} = b + \frac{v_0^2}{2g} + \frac{b^2 g}{2 v_0^2}}$$

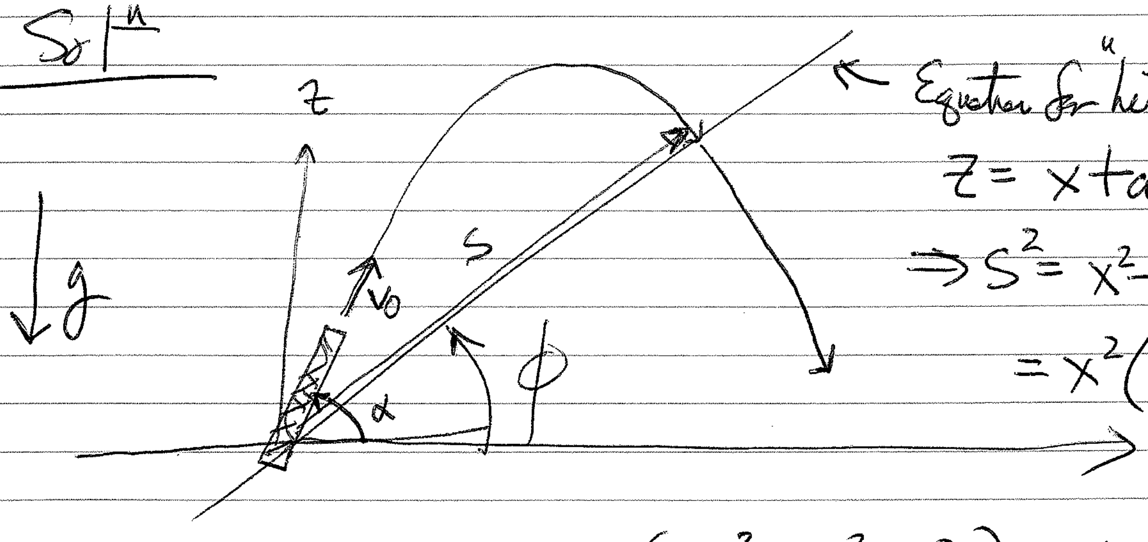
4.8 A gun is located at the bottom of a hill of constant slope ϕ. Show that the range of the gun measured up the slope of the hill is

$$\frac{2V_0^2 \cos\alpha \sin(\alpha-\phi)}{g\cos^2\phi}$$

where α is the $\measuredangle$ of elevation of the gun, and that the maximum value of the slope range is

$$\frac{V_0^2}{g(1+\sin\phi)}$$

Sol^n



← Equation for "hill";

$$z = x\tan\phi$$

$$\Rightarrow s^2 = x^2\tan^2\phi + x^2$$

$$= x^2(1+\tan^2\phi)$$

(i) find the maximum s ($s^2 = x^2 + z^2$) w/ the constraint

$$s = z/\sin\phi$$

&

$$s = x/\cos\phi$$

(ii) note: $\begin{cases} V_{0,x} = V_0\cos\alpha \\ V_{0,z} = V_0\sin\alpha \end{cases}$

(iii) Equation-of-Motion,

$$m\ddot{x} = 0 \rightarrow \dot{x} = V_{0,x} \rightarrow \boxed{x = V_{0,x} t}$$

$$m\ddot{z} = -mg \rightarrow \dot{z} = -gt + V_{0,z}$$

$$\rightarrow \boxed{z = -\tfrac{1}{2}gt^2 + V_{0,z} t}$$

The projectile stays in the air until

$$\frac{z}{x} = \tan\phi$$

$$\Longrightarrow \frac{V_0 \sin\alpha \cancel{t} - \frac{1}{2}gt^{\cancel{2}}}{V_0 \cos\alpha \cancel{t}} = \tan\phi$$

$$-\tfrac{1}{2}gt = V_0 \cos\alpha \tan\phi - V_0 \sin\alpha$$

$$\boxed{t_\phi = \frac{2V_0}{g}\left(\sin\alpha - \cos\alpha \tan\phi\right)}$$

(iv) the distance up the hill is then

$$S^2 = x^2(t_\phi) + z^2(t_\phi)$$

$$= (V_0 \cos\alpha)^2 t_\phi^2 + \left(V_0 \sin\alpha - \frac{g t_\phi}{2}\right)^2 t_\phi^2$$

$$\boxed{S^2 = V_0^2 t_\phi^2 - gV_0 \sin\alpha\, t_\phi^3 + \tfrac{1}{4}g^2 t_\phi^4}$$

$$t_\phi = \frac{2V_0 \cos\alpha}{g}\left(\tan\alpha - \tan\phi\right)$$

$$\Rightarrow s^2 = V_0^2\left(\frac{4V_0^2}{g^2}\left[\tan\alpha-\tan\phi\right]^2\cos^2\alpha\right)$$

$$- gV_0\sin\alpha\left(\frac{2V_0\cos\alpha}{g}\left[\tan\alpha-\tan\phi\right]\right)^3$$

$$+ \frac{g^2}{4}\left(\frac{2V_0\cos\alpha}{g}\left[\tan\alpha-\tan\phi\right]\right)^4$$

$$\frac{g^2}{4V_0^4}s^2 = \cos^2\alpha\left[\tan\alpha-\tan\phi\right]^2 - 2\sin\alpha\cos^3\alpha\left[\tan\alpha-\tan\phi\right]^3$$

$$+ \cos^4\alpha\left[\tan\alpha-\tan\phi\right]^4$$

$$= \left[\sin\alpha-\cos\alpha\tan\phi\right]^2 - 2\sin\alpha\left[\sin\alpha-\cos\alpha\tan\phi\right]^3$$

$$+ \left[\sin\alpha-\cos\alpha\tan\phi\right]^4$$

extremize $\left(\frac{gs}{2V_0^2}\right)^2$

$$\frac{\partial}{\partial\alpha}\left[\frac{g^2s^2}{4V_0^4}\right] = 2\left[\sin\alpha-\cos\alpha\tan\phi\right]\left(\cos\alpha+\sin\alpha\tan\phi\right)$$

$$- 2\cos\alpha\left[\sin\alpha-\cos\alpha\tan\phi\right]^3 - 6\sin\alpha\left[\sin\alpha-\cos\alpha\tan\phi\right]^2\left(\cos\alpha+\sin\alpha\tan\phi\right)$$

$$+ 4\left[\sin\alpha-\cos\alpha\tan\phi\right]^3\left(\cos\alpha+\sin\alpha\tan\phi\right)$$

$$= 0$$

factor out $(\sin\alpha-\cos\alpha\tan\phi)$ $\Rightarrow$ 1 root is $\boxed{\tan\alpha = \tan\phi}$

$$\left(\frac{gs}{2V_0^2}\right)^2 = (\sin\alpha - \cos\alpha\tan\phi)^2\left\{1 - 2\sin\alpha(\sin\alpha - \cos\alpha\tan\phi) + (\sin\alpha - \cos\alpha\tan\phi)^2\right\}$$

$$= (\sin\alpha - \cos\alpha\tan\phi)^2\left\{1 - \sin^2\alpha + \cos^2\alpha\tan^2\phi\right\}$$

$$= \cos^2\alpha(\sin\alpha - \cos\alpha\tan\phi)^2(1 + \tan^2\phi)$$

$$= \frac{\cos^2\alpha}{\cos^4\phi}\underbrace{(\sin\alpha\cos\phi - \cos\alpha\sin\phi)}^2$$

$$= \frac{\cos^2\alpha}{\cos^4\phi}(\sin[\alpha - \phi])^2$$

$$\Rightarrow \boxed{S = \frac{2V_0^2}{g}\frac{\cos\alpha}{\cos^2\phi}\sin(\alpha - \phi)}$$

Extremize s

or $\frac{\partial}{\partial\alpha}\left(\frac{g\cos^2\phi\, S}{2V_0^2}\right) = 0$

$$\frac{\partial}{\partial\alpha}\left[\frac{g\cos^2\phi\, s}{2V_0^2}\right] = \frac{\partial}{\partial\alpha}\left[\cos\alpha\sin(\alpha - \phi)\right]$$

$$= -\sin\alpha\sin(\alpha - \phi) + \cos\alpha\cos(\alpha - \phi)$$

$$= \cos\left[\alpha + (\alpha - \phi)\right]$$

$$= \cos\left[2\alpha - \phi\right] = 0 \implies 2\alpha - \phi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$\alpha = \frac{\phi}{2} + \frac{\pi}{4}$$

and

$$S = \frac{2V_0^2}{g\cos^2\phi}\left[\cos\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\sin\left(\frac{\phi}{2} + \frac{\pi}{4} - \phi\right)\right]$$

$$= \frac{2V_0^2}{g\cos^2\phi}\left[\cos\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\sin\left(-\frac{\phi}{2} + \frac{\pi}{4}\right)\right]$$

$$= \frac{2V_0^2}{g\cos^2\phi}\left[\left(\cos\frac{\phi}{2}\cos\frac{\pi}{4} - \sin\frac{\phi}{2}\sin\frac{\pi}{4}\right)\left(\sin -\frac{\phi}{2}\cos\frac{\pi}{4} + \cos -\frac{\phi}{2}\sin\frac{\pi}{4}\right)\right]$$

$$= \frac{2V_0^2}{g\cos^2\phi}\sin^2\frac{\pi}{4}\left[\left(\cos\frac{\phi}{2} - \sin\frac{\phi}{2}\right)\left(\sin\left(-\frac{\phi}{2}\right) + \cos\left(-\frac{\phi}{2}\right)\right)\right]$$

$$= \frac{V_0^2}{g\cos^2\phi}\left[\cos\frac{\phi}{2}\sin\left(+\frac{\phi}{2}\right) + \cos\frac{\phi}{2}\cos\left(+\frac{\phi}{2}\right) + \sin\frac{\phi}{2}\sin\left(+\frac{\phi}{2}\right) - \sin\frac{\phi}{2}\cos\frac{\phi}{2}\right]$$

$$= \frac{V_0^2}{g\cos^2\phi}\left[-2\cos\frac{\phi}{2}\sin\frac{\phi}{2} + \cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}\right]$$

$$= \frac{V_0^2}{g\cos^2\phi}\left[1 - 2\cos\frac{\phi}{2}\sin\frac{\phi}{2}\right]$$

$$= \frac{V_0^2}{g\cos^2\phi}\left[1 - \sin\phi\right] = \boxed{\frac{V_0^2}{g(1+\sin\phi)} = S_{max}}$$, finally!

4.14

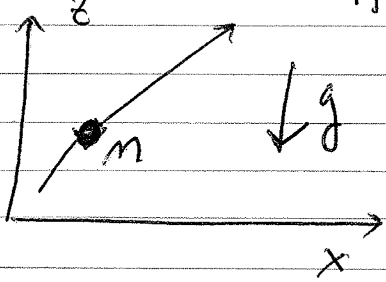
Write down the component form of the differential equations of motion of a projectile if the air resistance is proportional to the square of the speed. Are the equations separated? Show that the x-component of the velocity is

$$\dot{x} = \dot{x}_0 e^{-\gamma s}$$

where $s = \sqrt{x^2 + z^2}$ and $\gamma = c_2/m$

← if particle starts at ($x = z = 0$)

Sol^n



$$\boxed{\begin{aligned} m\ddot{x} &= -c_2(\dot{x}^2 + \dot{z}^2)^{\frac{1}{2}}\dot{x} \\ m\ddot{z} &= -mg - c_2(\dot{x}^2 + \dot{z}^2)^{\frac{1}{2}}\dot{z} \end{aligned}}$$

Integrate $\ddot{x} \Rightarrow \ddot{x} = -\frac{c_2}{m}(\dot{x}^2 + \dot{z}^2)^{\frac{1}{2}}\dot{x}$

$$\int \frac{d\dot{x}}{\dot{x}} = -\frac{c_2}{m}\underbrace{\int (\dot{x}^2 + \dot{z}^2)^{\frac{1}{2}} dt}_{s}$$

$$\ln\left(\frac{\dot{x}}{\dot{x}_0}\right) = -\frac{c_2}{m} s \Rightarrow \boxed{\dot{x} = \dot{x}_0 e^{-c_2 s/m}}$$

4.19

An atom is situated in a simple cubic crystal lattice. If the potential energy of interaction between 2 atoms is of the form

$$cr^{-\alpha}$$

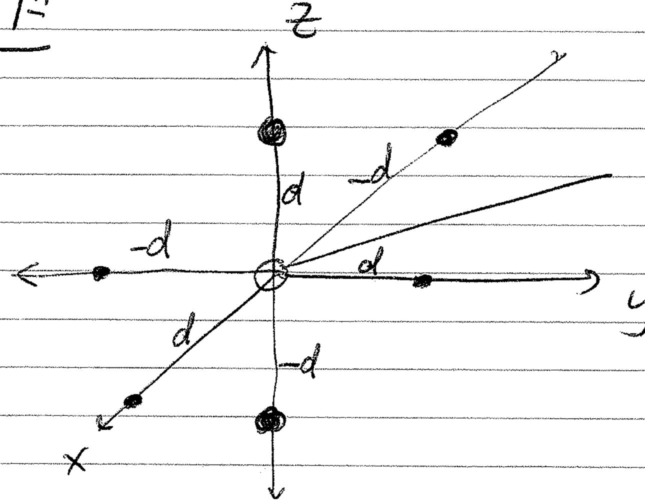
show that the total energy of the interaction of a given atom w/ its 6 nearest neighbors is approximately

$$V = A + B\left(x^2+y^2+z^2\right)$$

where A & B are constants. Assume that the 6 neighboring atoms are fixed and are located at the points

$$(\pm d,0,0),\ (0,\pm d,0),\ (0,0,\pm d)$$

Sol^{n}



let atom be located at

$$(0+x,\ 0+y,\ 0+z)$$

where $\sqrt{x^2+y^2+z^2} \ll d$

find V

$$V_x = C\left[\overbrace{\left(\underbrace{[d-x]^2}_{d^2\left(1-\frac{x}{d}\right)^2}+y^2+z^2\right)^{-\frac{\alpha}{2}} + \left(\underbrace{[-d-x]^2}_{d^2\left(1+\frac{x}{d}\right)^2}+y^2+z^2\right)^{-\frac{\alpha}{2}}}^{\text{particles on } x\text{-axis}}\right]$$

similar relations for particles on y- and z-axes

$$V_x \simeq C\left[\left(d^2-2xd+y^2+z^2\right)^{-\frac{\alpha}{2}}+\left(d^2+2xd+y^2+z^2\right)^{-\frac{\alpha}{2}}\right]$$

$$\simeq Cd^{-\alpha}\left[1+\left(\frac{y^2+z^2-2xd}{d^2}\right)\left(-\frac{\alpha}{2}\right)+1+\left(\frac{y^2+z^2+2xd}{d^2}\right)\left(-\frac{\alpha}{2}\right)\right]$$

$$= Cd^{-\alpha}\left[2-2\left(\frac{\alpha}{2}\right)\left(\frac{y^2+z^2}{d^2}\right)\right]$$

total energy is then $V_x+V_y+V_z$

$$\Rightarrow V = Cd^{-\alpha}\left[8-\frac{\alpha}{d^2}\left(y^2+z^2+x^2+z^2+x^2+y^2\right)\right]$$

$$= Cd^{-\alpha}\left[8-2\frac{\alpha}{d^2}\left(x^2+y^2+z^2\right)\right]$$

$$= 8Cd^{-\alpha}-2C\frac{\alpha}{d^{2+\alpha}}\left(x^2+y^2+z^2\right)$$

C, α are constants $\Rightarrow$ define $A = 8Cd^{-\alpha}$

$$B = -\frac{2C\alpha}{d^{2+\alpha}}$$

and then $\boxed{V = A + B\left(x^2+y^2+z^2\right)}$