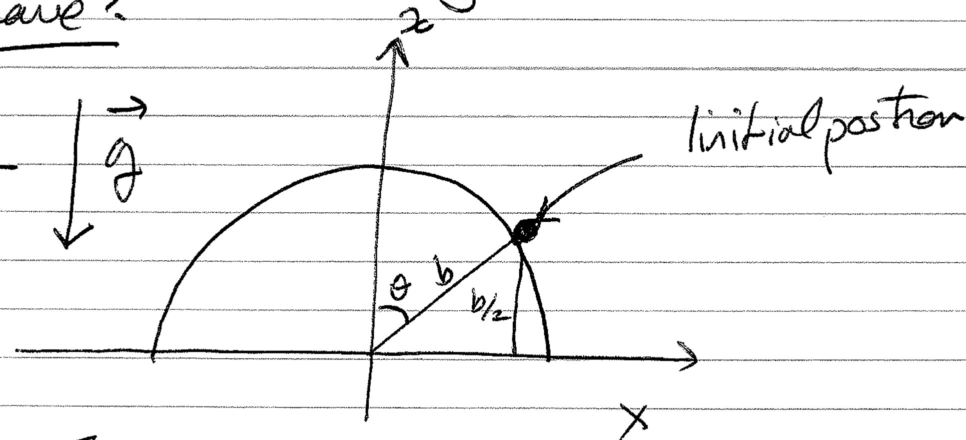
HW#2

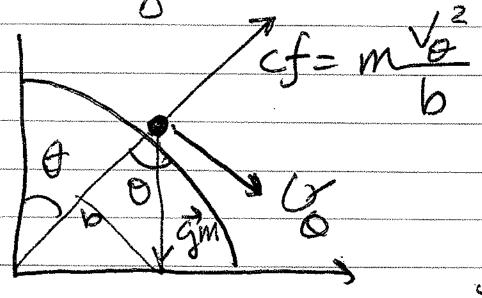
4.21

A particle is placed on a smooth sphere of radius b at a distance $b/2$ above the central plane. As the particle slides down the side of the sphere, at which point will it leave?

Sol^n



(i) To find where m leaves the sphere, find where the centrifugal force first overcomes gravity



$$\boxed{m\frac{v_\theta^2}{b} - mg\cos\theta > 0}$$

(ii) Find v_θ from the energy equation,

$$E = \frac{1}{2}mv_\theta^2 + mgz$$

at $t=0$, $v_\theta = 0$, $z = b/2$ $\Longrightarrow$ $E = \frac{mgb}{2} = \frac{mv_\theta^2}{2} + mgz$

and $\boxed{v_\theta^2 = bg - 2gz}$

(iii) $\Rightarrow \frac{m}{b}(bg - 2gz) = mg\cos\theta = mg\frac{z}{b}$

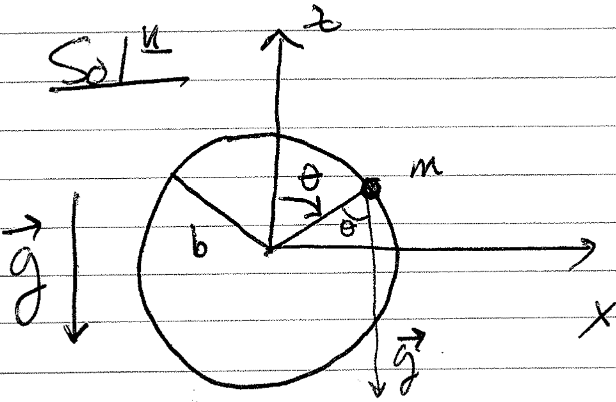
$$\Rightarrow \frac{z}{b} = 1 - 2\frac{z}{b}$$

$$\Rightarrow \boxed{\left(\frac{z}{b}\right) = \frac{1}{3}}$$

4.22

a bead slides on a smooth rigid wire bent into the form of a circular loop of radius b. If the plane of the loop is vertical, and if the bead starts from rest at a point that is level w/ the center of the loop, find the speed of the bead at the bottom and the reaction of the wire on the bead at that point

Sol^n



Equations-of-Motion

$$\begin{cases} F_r = -mg\cos\theta + m\frac{b^2\dot{\theta}^2}{b} + R \\ F_\theta = +mg\sin\theta \end{cases}$$

(a) Bead starts from $\theta = \pi/2$ w/ $V_\theta = 0$

from energy equation, $E = \frac{m}{2}V_\theta^2 + mgz = \frac{m}{2}V_\theta^2 + mgb\cos\theta$

$$\rightarrow E = 0 + 0 = 0$$

(b) Bottom of loop

$$E = 0 = \frac{m}{2}V_\theta^2 + mgb\cos(\pi) \Longrightarrow \boxed{V_\theta^2 = \cancel{2}\,2bg}$$

(c) Reaction comes from

$$F_r = -mg\cos\theta + m(V_\theta^2/b) + R$$

$$\Longrightarrow R = +mg\cos\theta - 2mg$$

$$\boxed{R_r = -3mg}$$

4.23

Show that the period of the particle sliding in the cycloid trough (Ex. 4.6.2) is $4\pi\sqrt{A/g}$.

Sol^n

(i) Consider Energy, $E = \frac{m}{2}\dot{s}^2 + \frac{1}{2}\left(\frac{mg}{4A}\right)s^2$

$$\frac{dE}{dt} = 0 = m\dot{s}\ddot{s} + \left(\frac{mg}{4A}\right)s\dot{s}$$

$$= \dot{s}m\left[\ddot{s} + \frac{g}{4A}s\right] = 0$$

$\dot{s} = 0 \Rightarrow$ trivial solution (s = constant)

and so, $\ddot{s} + \frac{g}{4A}s = 0$ is true in general

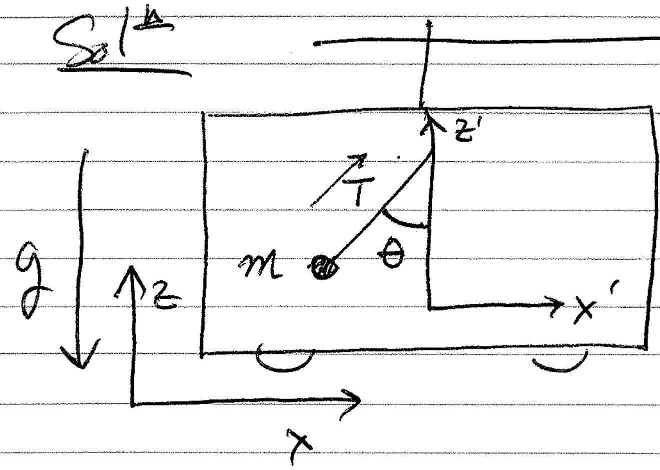
$$\Rightarrow \boxed{s = A\cos\sqrt{\frac{g}{4A}}\,t + B\sin\sqrt{\frac{g}{4A}}\,t}$$

and period of oscillation is $P = \frac{2\pi}{\sqrt{\frac{g}{4A}}} = 4\pi\sqrt{\frac{A}{g}}$

(ii) See next 2 pages for a discussion of the "Isochronous pendulum."

5.3 A plumb line is held steady while being carried along in a moving train. If the mass of the plumb bob is m, find the tension in the cord T and the deflection from vertical if the train accelerates forward w/ constant acceleration $g/10$. Ignore effects of Earth's rotation.

Sol^{n}



$$\vec{F} = -m\vec{g} + \vec{T} = m\left[\vec{a} + \ddot{\vec{r}}'\right]$$

acceleration of train ($\vec{a}$); tension in cord ($\vec{T}$); acceleration w/respect to train frame ($\ddot{\vec{r}}'$)

(i) for the non-inertial frame observer

$$m\ddot{\vec{r}}' = -m\vec{g} + \vec{T} - m\vec{a}$$

$$\Rightarrow \begin{cases} m\ddot{r}' = -mg\cos\theta + T - ma\sin\theta \\ m(l\ddot{\theta}') = -mg\sin\theta + ma\cos\theta \end{cases}$$

(ii) In equilibrium, $\ddot{r}' = 0$, $\ddot{\theta}' = 0$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = +\frac{a}{g} \Rightarrow T = m\left[g\cos\theta + a\sin\theta\right]$$

$$\left(\frac{\sin\theta}{\cos\theta}\right) = +\frac{1}{10} \qquad = mg\cos\theta\left[1 + \left(\frac{1}{10}\right)^2\right]$$

$$\boxed{\theta = \tan^{-1}\left(+\frac{1}{10}\right)} \qquad \boxed{T = 1.01\, mg\cos\theta}$$

5.4

Suppose the plumb executes small oscillations about equilibrium; find the period for the oscillation

Sol^n

$$\boxed{m l \ddot{\theta}' = -mg\sin\theta' + ma\cos\theta'} \leftarrow \text{from 5.3}$$

(i) perturb θ' as $\theta' = \underset{\uparrow \text{ equilibrium } \theta'}{\theta_0} + \delta\theta'$

$$\Rightarrow ml(\ddot{\delta\theta'}) = -mg\sin(\theta_0 + \delta\theta') + m\overset{g/10}{a}\cos(\theta_0 + \delta\theta')$$

$$= -mg\left[\sin\theta_0\cos\delta\theta' + \cos\theta_0\sin\delta\theta'\right]$$

$$+ ma\left[\cos\theta_0\cos\delta\theta' - \sin\theta_0\sin\delta\theta'\right]$$

for small $\delta\theta' \Rightarrow \cos\delta\theta' \simeq 1,\ \sin\delta\theta' \simeq \delta\theta'$

$$\simeq -mg\left[\sin\theta_0 + \delta\theta'\cos\theta_0\right] + ma\left[\cos\theta_0 - \delta\theta'\sin\theta_0\right]$$

$$\cancel{m} l\,\delta\ddot{\theta}' \simeq \delta\theta'\left[-\cancel{m}g\cos\theta_0 - \cancel{m}a\sin\theta_0\right] - \underbrace{\left[mg\sin\theta_0 - ma\cos\theta_0\right]}_{=0,\ \text{in equilibrium}}$$

$$\delta\ddot{\theta}' \simeq -\frac{g}{l}\left(\cos\theta_0 + \frac{1}{10}\sin\theta_0\right)\delta\theta'$$

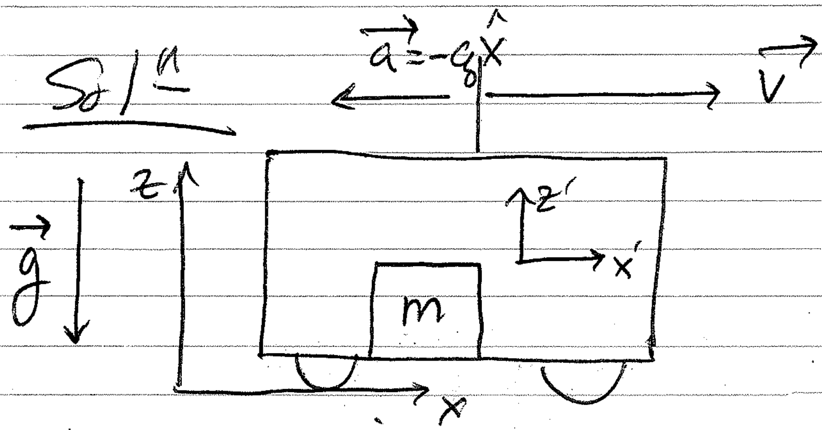
$$\Rightarrow \boxed{P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \Big/ \sqrt{\cos\theta_0 + \frac{\sin\theta_0}{10}}}$$

5.5

A hauling truck is traveling on a level road. The driver suddenly applies the brakes causing the truck to decelerate at rate $g/2$. This causes the box in the rear to slide forward. If the coefficient of sliding friction between the box and the truckbed $\mu = (1/3)$, find the acceleration relative to (a) the truck and (b) the road.

why not static friction?

Sol^{n}



ⓐ Equation-of-Motion — Noninertial Frame

(i) $F_z = m\ddot{z}' = -mg + N = 0 \Rightarrow N = mg$

(ii) $F_x = -\mu mg = m\left[-a_0 + \ddot{x}'\right]$

$$\Rightarrow \boxed{\ddot{x}' = a_0 - \mu g = +g/2 - g/3 = g/6}$$

truck frame

ⓑ Inertial Frame

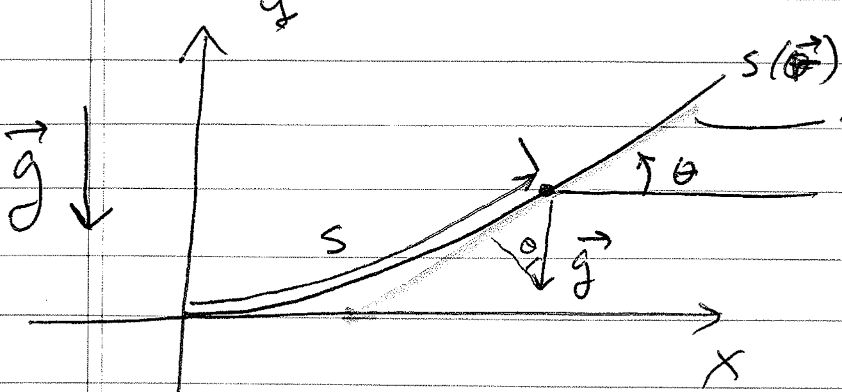
(i) $\ddot{\vec{r}} = \ddot{\vec{R}}_0 + \ddot{\vec{r}}' = -\frac{g}{2}\hat{x} + \frac{g}{6}\hat{x}'$

$\hat{x} // \hat{x}'$

$$\Rightarrow \boxed{\ddot{\vec{r}} = -\frac{1}{3} g\hat{x}}$$

Example, "Isochronous Pendulum"

Can we form a surface (slope), such that if an object rolls along the surface, it will oscillate w/ period independent of amplitude?



tangent to $s(\vec{x})$ at s

$\Rightarrow$ θ is the $\angle$ between the horizontal and that the tangent

(i) force along the direction of motion is then

$$-mg\sin\theta$$

$$\Rightarrow \boxed{m\ddot{s} = -mg\sin\theta}$$

(ii) to have isochronous (harmonic motion), we must have that the force be a Hooke's law force

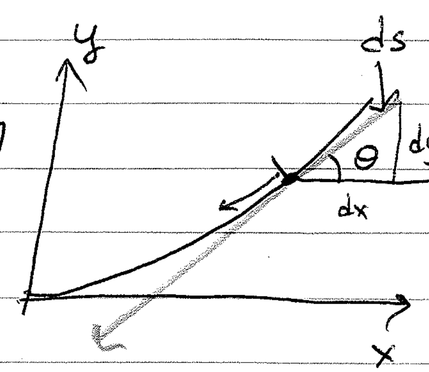
$$\Rightarrow m\ddot{s} = -ks$$

or $\boxed{-ks = -mg\sin\theta}$ $\rightarrow s = \frac{mg}{k}\sin\theta$

(iii) we have that

$$\frac{dx}{d\theta} = \frac{dx}{ds}\frac{ds}{d\theta} = \cos\theta \times \left[\frac{mg}{k}\cos\theta\right]$$

$$\boxed{\frac{dx}{d\theta} = \frac{mg}{k}\cos^2\theta}$$



(iv) We also have

$$\frac{dy}{d\theta} = \frac{dy}{ds}\frac{ds}{d\theta} = \sin\theta \times \left[\frac{mg}{k}\cos\theta\right]$$

$$\boxed{\frac{dy}{d\theta} = \frac{mg}{k}\sin\theta\cos\theta}$$

(v) <u>Solve for $y(\theta)$ & $x(\theta)$</u>

$$\Rightarrow x(\theta) = \frac{mg}{k}\int \cos^2\theta \, d\theta$$

note: $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\cos(\theta \pm \theta) = \cos^2\theta \mp \sin^2\theta$$

$$= \cos^2\theta \mp (1 - \cos^2\theta)$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\Rightarrow \boxed{x(\theta) = \frac{mg}{k}\int\left[1 + \cos 2\theta\right]d\theta = \frac{mg}{k}\left[\theta + \frac{1}{2}\sin 2\theta\right]}$$

$$\Rightarrow y(\theta) = \frac{mg}{k}\int \sin\theta\cos\theta \, d\theta$$

$$= \frac{mg}{k}\int \sin\theta \, d(\sin\theta)$$

$$= \frac{mg}{2k}\sin^2\theta$$

$$\boxed{y(\theta) = -\frac{mg}{4k}\cos 2\theta}$$

"Cycloid"