HW #3

5.6

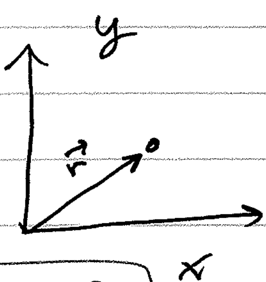
The position of a particle in a fixed frame of reference is given by

$$\vec{r} = \hat{i}\left(x_0 + R\cos\Omega t\right) + \hat{j}\, R\sin\Omega t$$

where x_0, R, & Ω are constants.

a) Show that the particle moves in a circle w/ constant speed.

(i) find the trajectory given that



$$\begin{cases} x = x_0 + R\cos\Omega t \\ y = R\sin\Omega t \end{cases}$$

$$\Rightarrow \boxed{(x-x_0)^2 + y^2 = R^2\left[\cos^2\Omega t + \sin^2\Omega t\right] = R^2}$$

$\Rightarrow$ a circle with center $(x_0, 0)$ and radius R

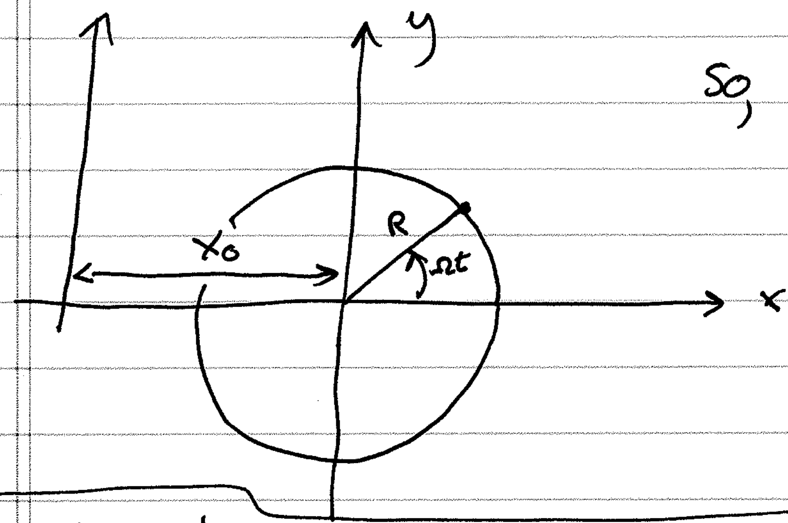
(ii) find speed using $\dot{\vec{r}} = \hat{i}\left(-\Omega R\sin\Omega t\right) + \hat{j}\,\Omega R\cos\Omega t$

$$\Rightarrow \dot{x}^2 + \dot{y}^2 = R^2\Omega^2\left[\sin^2\Omega t + \cos^2\Omega t\right] = R^2\Omega^2$$

$$\Rightarrow \boxed{V^2 = (\Omega R)^2 = \text{constant}}$$

(iii) Find 2 coupled, first-order differential equations that relate the components of position x' & y' and the components of velocity $\dot{x}'$ & $\dot{y}'$ of the particle in a frame rotating w/ frequency $\vec{\omega} = k\omega$

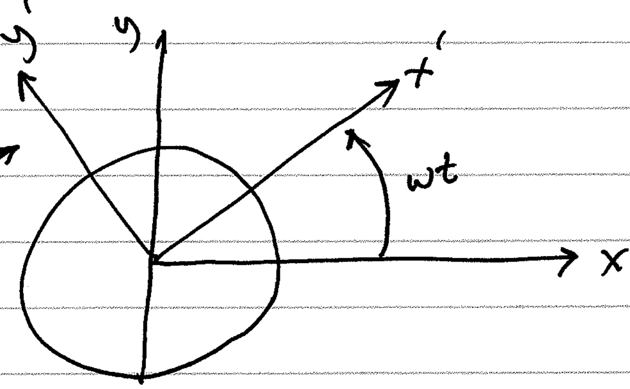
Let's change the problem. Move the origin to the center of the orbit (move origin $+x_0$ to the right).



So, $\vec{r}$ becomes

$$\vec{r} = \hat{\imath} R\cos\Omega t + \hat{\jmath} R\sin\Omega t$$

and let us define $\vec{r}\,'$ as



$$\Rightarrow \ddot{\vec{r}} = \cancel{\ddot{\vec{R}}} + \ddot{\vec{r}}\,' + 2\vec{\omega}\times\dot{\vec{r}}\,' + \dot{\vec{\omega}}\times\vec{r}\,' + \vec{\omega}\times(\vec{\omega}\times\vec{r}\,')$$

$$= \ddot{\vec{r}}\,' + 2(0,0,\omega)\times(\dot{x}',\dot{y}',0) + (0,0,\omega)\times(0,0,\omega)\times(x',y',0)$$

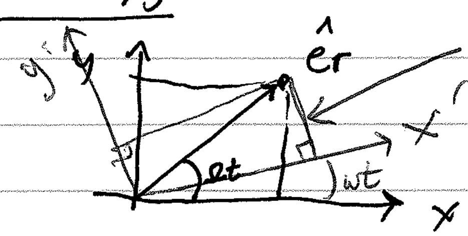
$$= \ddot{\vec{r}}\,' + 2(-\omega\dot{y}',\omega\dot{x}',0) + (0,0,\omega)\times(-\omega y',\omega x',0)$$

$$= \ddot{\vec{r}}\,' + 2(-\omega\dot{y}',\omega\dot{x}',0) + (-\omega^2 x',-\omega^2 y',0)$$

$$\ddot{\vec{r}} = \hat{\imath} R(-\Omega^2\cos\Omega t) + \hat{\jmath} R(-\Omega^2\sin\Omega t)$$ from above def^{n} of $\vec{r}$

$$= -\Omega^2 R\hat{r}$$

Relate $\hat{r}$ to $\hat{\imath}', \hat{\jmath}'$



$\hat{\imath}'\cos(\Omega-\omega)t + \hat{\jmath}'\sin(\Omega-\omega)t$

Equation-of-Motion $\vec{\ddot{r}}\cdot\hat{x}'$

$$\boxed{\begin{aligned}\ddot{x}' &= -\Omega^2 R\cos(\Omega-\omega)t - 2(-\omega\dot{y}') - (-\omega^2 x')\\ \ddot{y}' &= -\Omega^2 R\sin(\Omega-\omega)t - 2(\omega\dot{x}') - (-\omega^2 y')\end{aligned}}$$

find u

multiply $\ddot{y}'$ by i and add to $\ddot{x}'$

$$\underbrace{(\ddot{x}'+i\ddot{y}')}_{\ddot{u}} = -\Omega^2 R^*\left[\cos(\Omega-\omega)t + i\sin(\Omega-\omega)t\right] + 2\omega\underbrace{(\dot{y}'-i\dot{x}')}_{i\dot{u}} + \omega^2(x'+iy')$$

$$\ddot{u} = -\Omega^2 R^* e^{i(\Omega-\omega)t} - 2i\omega\dot{u} + \omega^2 u$$

$$\rightarrow \ddot{u} + 2i\omega\dot{u} - \omega^2 u = -\Omega^2 R^* e^{i(\Omega-\omega)t}$$

Solve by trying $u = a e^{i(\Omega-\omega)t}$

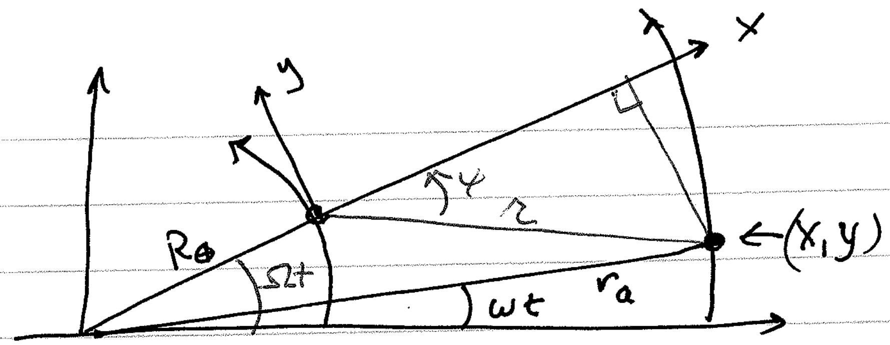
$$\rightarrow -(\Omega-\omega)^2 a\cancel{e^{i(\Omega-\omega)t}} - 2\omega(\Omega-\omega)a\cancel{e^{i(\Omega-\omega)t}} - \omega^2 a\cancel{e^{i(\Omega-\omega)t}} = -\Omega^2 R\cancel{e^{i(\Omega-\omega)t}}$$

$$\Rightarrow a = \frac{-\Omega^2 R}{-(\Omega-\omega)^2 - 2\omega(\Omega-\omega) - \omega^2} = R$$

and $u = Re^{i(\Omega-\omega)t} = x' + iy'$

$$\Longrightarrow \boxed{\begin{aligned}x' &= R\cos(\Omega-\omega)t\\ y' &= R\sin(\Omega-\omega)t\end{aligned}}$$

5.2



a) $r^2 = R_\oplus^2 + r_a^2 - 2R_\oplus r_a \cos\left[(\Omega-\omega)t\right]$

$$x(t) = r\cos\psi \quad \& \quad y(t) = -r\sin\psi$$

what is $\cos\psi$?

from law of cosines

$$r_a^2 = R_\oplus^2 + r^2 - 2R_\oplus r \overbrace{\cos(\pi-\psi)}^{-\cos\psi}$$

$$\to \cos\psi = \frac{r_a^2 - R_\oplus^2 - r^2}{2R_\oplus r}$$

$-2R_\oplus^2 + 2R_\oplus r_a \cos(\Omega-\omega)t$

$$\Longrightarrow \boxed{\begin{aligned} x(t) &= \frac{1}{2R_\oplus}\left(r_a^2 - R_\oplus^2 - r^2\right) \\ y(t) &= \frac{-1}{2R_\oplus}\left(4R_\oplus^2 r^2 - \left[r_a^2 - R_\oplus^2 - r^2\right]^2\right)^{\frac{1}{2}} \end{aligned}}$$

substitute for r^2 and write $x(t)$, $y(t)$ in terms of $R_\oplus, r_a, \Omega, \omega, t$.

a) $$x(t) = \frac{1}{2R_\oplus}\left[r_a^2 - R_\oplus^2 - r^2\right]$$

$$= \frac{1}{2R_\oplus}\left[\cancel{r_a^2} - R_\oplus^2 - \left(R_\oplus^2 + \cancel{r_a^2} - 2R_\oplus r_a \cos(\Omega-\omega)t\right)\right]$$

$$= +\frac{1}{2R_\oplus}\left[-2R_\oplus^2 + 2R_\oplus r_a \cos(\Omega-\omega t)\right]$$

$$\boxed{x(t) = -R_\oplus + r_a \cos(\Omega-\omega)t}$$

b) $$y(t) = \frac{\pm 1}{2R_\oplus}\sqrt{4R_\oplus^2 r^2 - \left(r_a^2 - R_\oplus^2 - r^2\right)^2}$$

$$= \pm\sqrt{r^2 - \left(\frac{r_a^2}{2R_\oplus} - \frac{R_\oplus}{2} - \frac{r^2}{2R_\oplus}\right)^2}$$

$$= \pm\sqrt{\left(R_\oplus^2 + r_a^2 - 2R_\oplus r_a \cos(\Omega-\omega)t\right) - \left(-R_\oplus + r_a \cos(\Omega-\omega)t\right)^2}$$

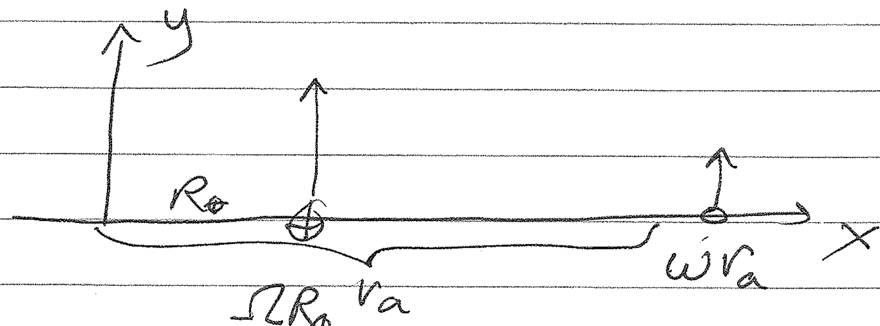
$$= \pm\sqrt{\left(\cancel{R_\oplus^2} + r_a^2 - \cancel{2R_\oplus r_a \cos}(\Omega-\omega)t\right) - \left(\cancel{R_\oplus^2} + r_a^2 \cos^2(\Omega-\omega)t - \cancel{2R_\oplus r_a \cos}(\Omega-\omega)t\right)}$$

$$= \pm\sqrt{r_a^2\left(1-\cos^2[\Omega-\omega]t\right)}$$

$$= \pm r_a \sin(\Omega-\omega)t$$

$$\boxed{y(t) = -r_a \sin(\Omega-\omega)t}$$

b) Find $V_{asteroid}$ in ⊕'s frame at $t = 0$



$$\boxed{\begin{aligned} V_y &= (\omega r_a - \Omega R_\oplus) < 0 \\ V_{\cancel{x}\,y} &= 0 \end{aligned}}$$

c) Find the acceleration & equation-of-motion of the asteroid in the ⊕'s frame.

$$\ddot{\vec{r}}_{Lab} = \ddot{\vec{R}}_0 + \ddot{\vec{r}} + 2\vec{\Omega}\times\dot{\vec{r}} + \cancelto{0}{\dot{\vec{\Omega}}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

lab-frame (stationary frame)

translation of ⊕'s frame

acceleration relative to ⊕

$$-\omega^2 r_a \hat{r}_a = \overbrace{-\Omega^2 R_\oplus \hat{R}_\oplus}^{-\Omega^2 R_\oplus \hat{x}} + \ddot{\vec{r}} + 2\vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times\left[\vec{\Omega}\times\vec{r}\right]$$

$$\hat{r}_a = \hat{x}\cos(\Omega t-\omega t) - \hat{y}\sin(\Omega t-\omega t)$$

<u>write down components of acceleration</u>

$$\ddot{x} = -\omega^2 r_a \cos\left[(\Omega-\omega)t\right] + \Omega^2 R_\oplus + 2\Omega\dot{y} + \Omega^2 x - \cancel{\omega^2 R_\oplus}$$

$$\ddot{y} = +\omega^2 r_a \sin\left[(\Omega-\omega)t\right] - 2\Omega\dot{x} + \Omega^2 y$$

multiply $\ddot{y}$ by i & add

$$\ddot{x} + i\ddot{y} = -\omega^2 r_a \cos\left[(\Omega-\omega)t\right] + i\omega^2 r_a \sin(\Omega-\omega)t + \Omega^2 R_\oplus - \cancel{\omega^2 R_\oplus} + 2\Omega(\dot{y}-i\dot{x}) + \Omega^2(x+iy)$$

$$\Rightarrow \ddot{u} = -\omega^2 r_a \exp\left[i(\omega-\Omega)t\right] + \Omega^2 R_\oplus - \cancel{\omega^2 R_\oplus} - 2i\Omega\dot{u} + \Omega^2 u$$

and

$$\underbrace{\ddot{u} + 2i\Omega\dot{u} - \Omega^2 u + \omega^2 r_a e^{i(\omega-\Omega)t}} = (\Omega^2 - \cancel{\omega^2})R_\oplus$$

treat this as homogeneous part, otherwise define

$$U = u + \left(\frac{\Omega^2 - \cancel{\omega^2}}{\Omega^2}\right)R_\oplus \rightarrow \dot{U} = \dot{u} \rightarrow \ddot{U} = \ddot{u}$$

$$\Rightarrow \ddot{U} + 2i\Omega\dot{U} - \Omega^2 U + \omega^2 r_a e^{i(\omega-\Omega)t} = 0$$

<u>Let</u> $U = a e^{i(\omega-\Omega)t}$, substitute in and then cancel $e^{i(\omega-\Omega)t}$ terms

$$\Rightarrow -(\omega-\Omega)^2 a - 2\Omega(\omega-\Omega)a - \Omega^2 a + \omega^2 r_a = 0$$

$$\rightarrow \quad a = \frac{+\omega^2 r_a}{(\omega-\Omega)^2 + 2\Omega(\omega-\Omega) + \Omega^2}$$

$$= +r_a \checkmark$$

and

$$v = +r_a e^{i(\omega-\Omega)t}$$

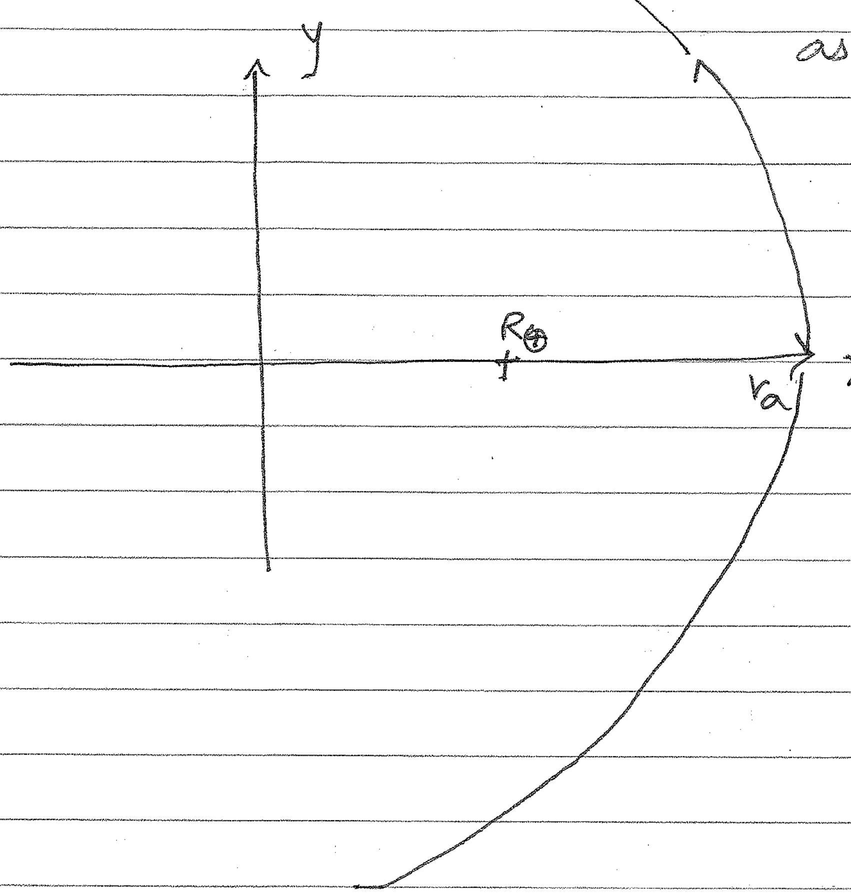
$$= x + iy + R_\oplus$$

equate real parts & imaginary parts

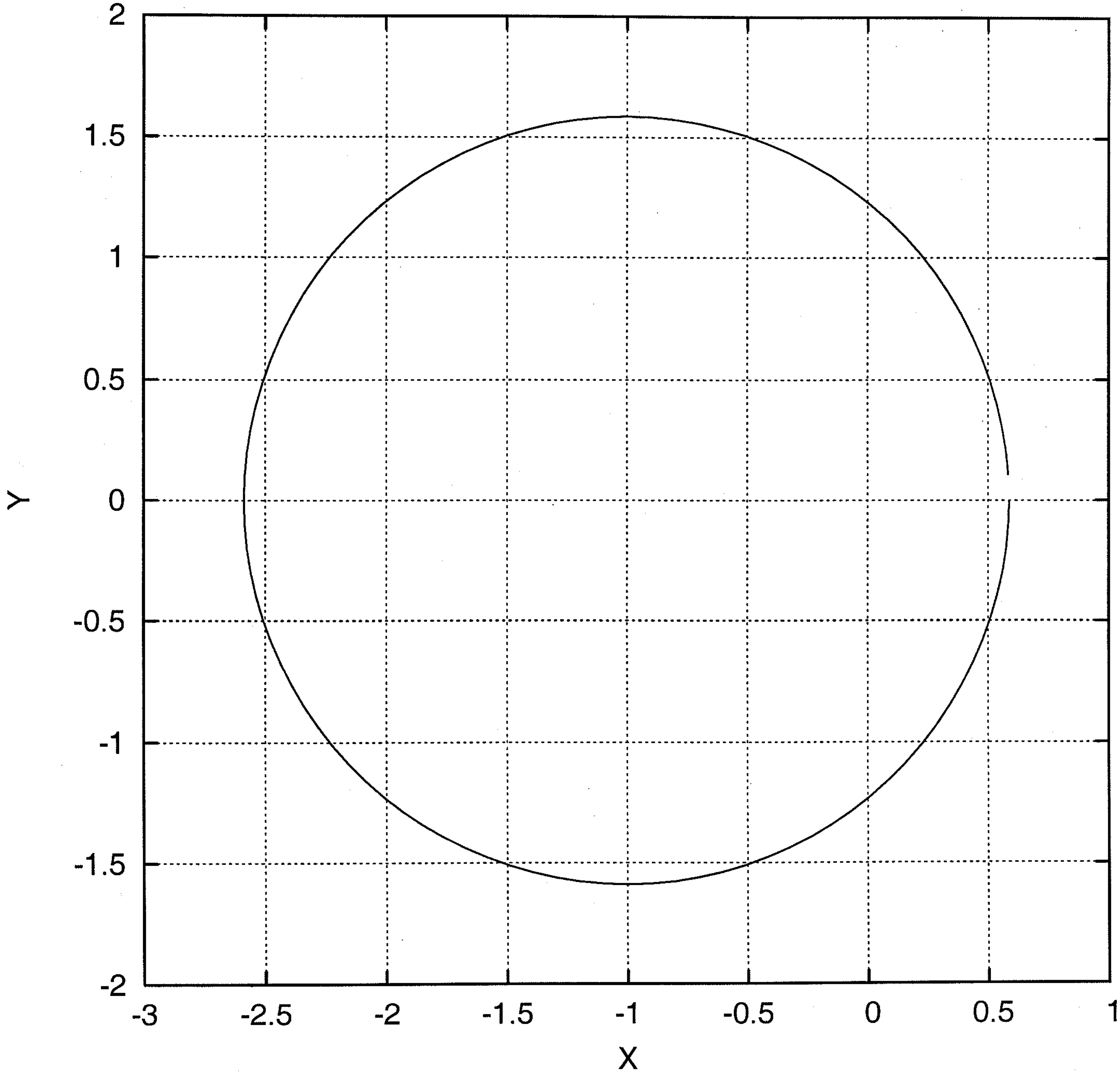
$$\boxed{\begin{aligned} x &= +r_a \cos(\omega-\Omega)t - R_\oplus = r_a \cos(\Omega-\omega)t - R_\oplus \\ y &= +r_a \sin(\omega-\Omega)t = -r_a \sin(\Omega-\omega)t \end{aligned}}$$

as asked.

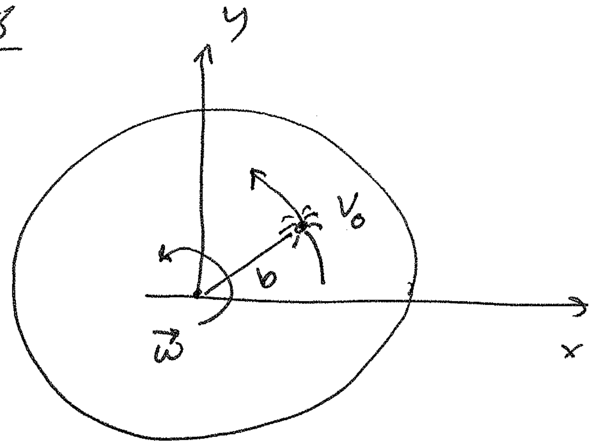
d)



r_a orbit is an offset circle in the frame of the ⊕


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
Y
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
X

5.8



Cockroach crawls in a circular path (radius b) w/speed relative to the turntable of v_0.

a) find v_0 to make the cockroach slip if he moves in the direction of spin,

b) if he moves opposite to the spin

Solution

a) $\ddot{\vec{r}} = \ddot{\vec{R}}_0 + \ddot{\vec{r}}' + 2\vec{\omega}\times\dot{\vec{r}}' + \dot{\vec{\omega}}\times\vec{r}' + \vec{\omega}\times[\vec{\omega}\times\vec{r}']$

define a rotating frame with $\vec{\Omega}_F = (\omega + \frac{v_0}{b})\hat{z}$ so that the cockroach is stationary in the frame and located at $x' = b, y' = 0$

$$\Rightarrow \ddot{\vec{r}} = \cancel{\ddot{\vec{R}}_0}^{\,0} + \ddot{\vec{r}}' + 0 + 0 + \vec{\Omega}_f\times(\vec{\Omega}_f\times b\hat{x}')$$

$$= \ddot{\vec{r}}' + (0,0,\Omega_f)\times[0,\Omega_f b, 0]$$

$$= \ddot{\vec{r}}' + (-\Omega_f^2 b, 0, 0)$$

static friction coefficient ~~viscosity~~ cockroach mass

$\ddot{\vec{r}}' = \ddot{\vec{r}} + \Omega_f^2 b\hat{x}'$, cockroach slips when $+\mu mg - \Omega_f^2 b = 0$

or $\mu mg - [\omega + \frac{v_0}{b}]^2 b = \mu mg - (\omega^2 + \frac{v_0^2}{b^2} + 2\omega\frac{v_0}{b})b = 0$

$$\to + \frac{v_0^2}{b} + 2\omega v_0 \overset{-}{\cancel{*}}(\mu mg - \omega^2 b) = 0$$

$$v_0 = \frac{-2\omega \pm \sqrt{4\omega^2 + 4(\frac{\mu mg}{b} - \omega^2)}}{\frac{2}{b}}$$

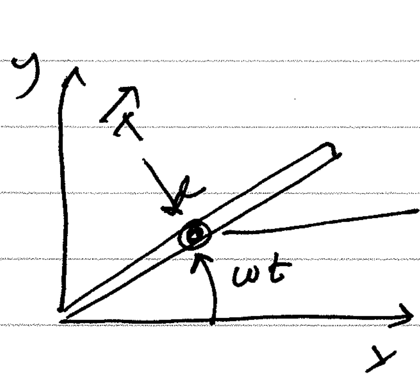
$$= (-\omega \pm \sqrt{\frac{\mu mg}{b}})b = (\sqrt{\frac{\mu mg}{b}} - \omega)b$$

b) Opposite to spin $\Rightarrow \Omega_f = (\omega - v_0/b)$

$$\Rightarrow \mu mg - b(\omega^2 + \frac{v_0^2}{b^2} - 2\omega\frac{v_0}{b}) = 0$$

$$\to v_0 = (\cancel{2}\omega \pm \sqrt{\frac{\mu mg}{b}})b = (\omega + \sqrt{\frac{\mu mg}{b}})b$$

5.10



at $t=0$, bead (mass m) is released from rest at $\ell/2$ (midpoint of rod).

Solution

a) find the displacement as a function of time (assumes motion along the rod).

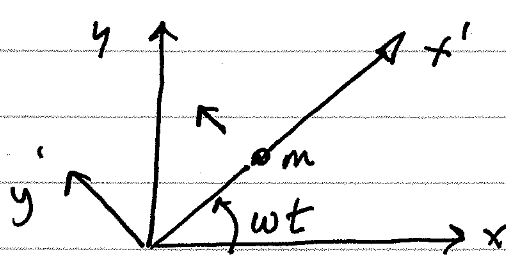
$$\ddot{\vec r} = \ddot{\vec R}_0 + \ddot{\vec r}\,' + 2\vec\omega\times\dot{\vec r}\,' + \dot{\vec\omega}\times\vec r\,' + \vec\omega\times\left[\vec\omega\times\vec r\,'\right]$$

No force translation (only rotation) $\rightarrow \ddot{\vec R}_0 = 0$

acceleration in lab frame is due to reaction force $\vec T$, $\perp$ to rod

So, for motion along the rod

$$0 = 0 + \ddot{\vec r}\,' + 2\vec\omega\times\dot{\vec r}\,' + 0 + \vec\omega\times\left[\vec\omega\times\vec r\,'\right]$$



$$m\ddot x' = +\underbrace{0}_{\perp \text{ to rod}} + m\omega^2 x'$$

$$\frac{d}{dt}\left(\frac{dx'}{dt}\right) = \frac{\omega^2}{\cancel{m}}x'$$

let $P = \frac{dx'}{dt} \Rightarrow \frac{d}{dt}P = \frac{dP}{dx'}\frac{dx'}{dt} = P\frac{dP}{dx'}$

$$P\frac{dP}{dx'} = \frac{\omega^2}{\cancel{m}}x' \Rightarrow \frac{P^2}{2}\Big|_0^{\dot x'} = \frac{\omega^2}{\cancel{m}}\left(x'^2 - \frac{\ell^2}{4}\right)$$

$$\Rightarrow \frac{\dot x'^2}{2} = 0 + \frac{\omega^2}{\cancel{m}}x'^2 - \frac{\omega^2\ell^2}{4\cancel{m}}$$

$$\frac{dx'}{\left[\frac{\omega^2}{\cancel{m}}x'^2 - \frac{\omega^2 l^2}{4\cancel{m}}\right]^{1/2}} = \sqrt{2}\,dt'$$

$$\frac{\sqrt{\cancel{m}}}{\omega}\frac{dx'}{\sqrt{x'^2 - \frac{l^2}{4}}} = \sqrt{2}\,dt'$$

$$\sqrt{\frac{\cancel{m}}{\omega^2}}\ln\left[x' + \sqrt{x'^2 - \frac{l^2}{4}}\right]_{\frac{l}{2}}^{x'} = \sqrt{2}\,t'$$

$$\ln\left[\frac{x' + \sqrt{x'^2 - l^2/4}}{l/2}\right] = \omega\sqrt{\frac{2}{\cancel{m}}}\,t'$$

$$x' + \sqrt{x'^2 - l^2/4} = \frac{l}{2}e^{\omega\sqrt{\frac{2}{\cancel{m}}}t'}$$

$$\cancel{x'^2} - \frac{l^2}{4} = \left(\frac{l}{2}e^{\sqrt{\frac{2}{\cancel{m}}}\omega t}\right)^2 + \cancel{x'^2} - \cancel{2}x'\frac{l}{\cancel{2}}e^{\sqrt{\frac{2}{\cancel{m}}}\omega t}$$

$$x' = \frac{l}{4}e^{-\sqrt{\frac{2}{\cancel{m}}}\omega t} + \frac{l}{4}e^{\sqrt{\frac{2}{\cancel{m}}}\omega t}$$

$$\boxed{x' = \frac{l}{2}\cosh\sqrt{\frac{2}{\cancel{m}}}\,\omega t}$$

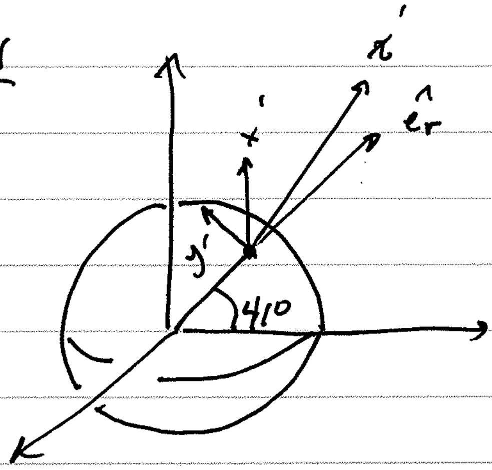
b) $\boxed{\dot{x}' = \frac{l}{2}\sqrt{\frac{2}{\cancel{m}}}\,\omega\sinh\sqrt{\frac{2}{\cancel{m}}}\,\omega t}$

c) beads leaves rod when $x' = l$

$\rightarrow \cosh\sqrt{\frac{2}{\cancel{m}}}\,\omega t_\infty = 2$ (note $\sinh^2 u = \cosh^2 u - 1$)

$\rightarrow \boxed{\dot{x}'_0 = \frac{\omega l}{\sqrt{2\cancel{m}}}3}$

5.11



at $t=0$

(i) $\dot{x}' = \dot{z}' = 0$

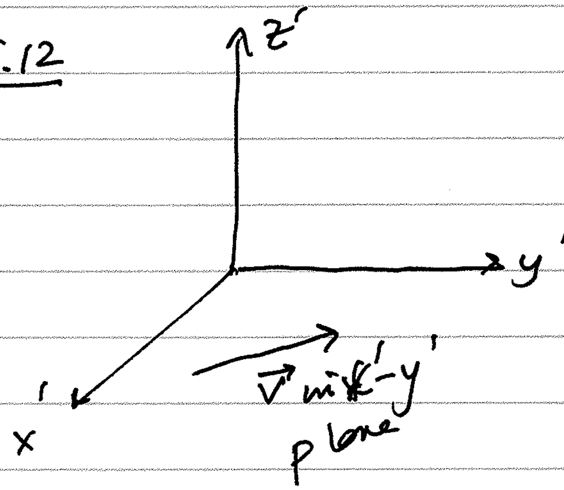
(ii) $\dot{y}' = 400$ mph $\approx 1.8\times10^4 \frac{cm}{s}$

$\omega_\oplus \simeq 7.3\times10^{-5}\frac{R}{s}$

a) $\frac{|F_{coriolis}|}{|weight|} = \frac{|\cancel{m}\,\omega_\oplus \dot{y}'|}{|\cancel{m}\, g_{eff}|} \simeq \frac{1.3\ \frac{cm}{s^2}}{980\ \frac{cm}{s^2}} \simeq 1.3\times10^{-3}$

b) $\vec{\omega}\times\dot{\vec{y}}'|$ $\Rightarrow -\hat{x}'$ direction

5.12



$\vec{\omega}\times\dot{\vec{r}}'$

$= \underbrace{(0, \omega_{y'}, \omega_{z'})}_{x' \text{ is in direction of spin rotation (E)}} \times (\dot{x}', \dot{y}', 0)$

$= (-\omega_{z'}\dot{y}', \omega_{z'}\dot{x}', -\omega_{y'}\dot{x}')$

In horizontal plane, $x'-y'$ plane, magnitude of acceleration is

$\sqrt{\omega_{z'}^2 \dot{y}'^2 + \omega_{z'}^2 \dot{x}'^2} = \omega_{z'}\sqrt{\dot{y}'^2 + \dot{x}'^2}$

$= \omega_{z'} V \leftarrow$ magnitude of $\vec{V}'$