HW #6

7.1

a) $\vec{R}_{CM} = \sum_{j=1}^{3} m_j \vec{r_j} \Big/ \sum_{j=1}^{3} m_j$

$\Rightarrow X_{CM} = 1/3$

$Y_{CM} = 2/3$

$Z_{CM} = 2/3$

b) $\vec{V}_{CM} = \sum_{j=1}^{3} \dot{\vec{R}}_{CM} = \sum_{j=1}^{3} m_j \dot{\vec{r}}_j \Big/ \sum_{j=1}^{3} m_j$

$\Rightarrow \dot{X}_{CM} = 3/3 = 1$

$\dot{Y}_{CM} = 2/3 = 2/3$

$\dot{Z}_{CM} = 1/3 = 1/3$

c) $\sum_{j=1}^{3} m_j \vec{v}_j \Rightarrow P_x = 2$

$P_y = 2$

$P_z = 1$

7.2

a) T of system in 7.1, $T = \frac{1}{2}\left[8\right] = 4$

b) find $\frac{M}{2} V_{cm}^2 = \frac{1}{2} \times 3 \times \frac{5}{3} = 5/2$

c) find $\vec{L}$ about the axis

$$\vec{L} = \sum_{j=1}^{3} \vec{r_j} \times m_j \vec{v_j} = \left[(-2\hat{k}) + (-\hat{i}) + (\hat{j} - \hat{i})\right]$$

$$\Rightarrow \vec{L} = -2\hat{i} + 1\hat{j} - 2\hat{k}$$

1.3

a bullet of mass m is fired from a gun of mass M. If the muzzle velocity of the bullet is V_0, show that the actual velocity of the bullet relative to the ground is

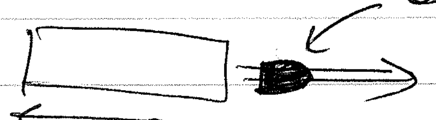
$$\frac{V_0}{1+\gamma}$$

and that the recoil velocity for the gun is

$$-\gamma \frac{V_0}{1+\gamma}$$

where $\gamma = mM$

Sol^{n}

a) 

leaves muzzle w/ V_0 relative to gun

V_R, gun recoils w/ V_R relative to the ground

b) $V_{grad} = V_0 - V_R$, if we define $V_R > 0$

c) Center-of-Mass remains fixed

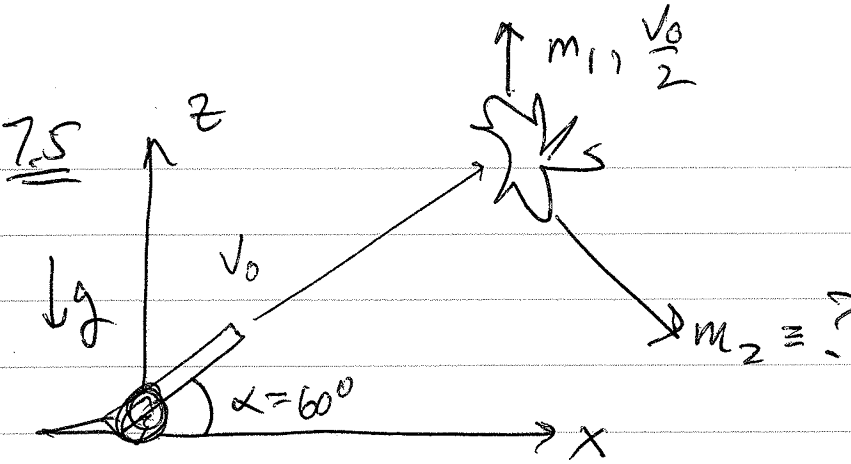
$$\Rightarrow m V_{grad} = M V_R$$

and so,

$$V_{grad} = V_0 - \frac{m}{M} V_{grad} \Rightarrow \boxed{V_{grad} = \frac{V_0}{1+\frac{m}{M}}}$$

d) $\Rightarrow \boxed{V_R = \frac{m}{M}\frac{V_0}{1+(M/m)}}$

7.5



a) Equation of Motion for the shell:

(i) $m\ddot{z} = -mg \longrightarrow \dot{z} = V_0 \sin\alpha - gt$

$z = V_0 \sin\alpha\, t - \frac{1}{2} g t^2$

(ii) $m\ddot{x} = 0 \longrightarrow \dot{x} = V_0 \cos\alpha$

$x = V_0 \cos\alpha\, t$

b) at peak, $\dot{z} = 0 \Longrightarrow \boxed{t_{peak} = V_0 \sin\alpha / g}$. "Patlo"

properties at t_{peak} are

(i) $\dot{z} = 0, \quad \dot{x} = V_0 \cos\alpha$

(relevant properties that is)

c) Conserve momentum & energy (?)

(i) $P_x = M\dot{x}$ & $P_z = M\dot{z}$ at "burst"

(ii) $\Rightarrow P_z = 0 \Rightarrow m_1\left(\frac{V_0}{2}\right)\uparrow$ and so, we must have

$(M - m_1)V \downarrow$ cancel P_z up

$\Rightarrow m_1\left(\frac{V_0}{2}\right) = (M - m_1)V$

$$\boxed{V_z = \left(\frac{m_1}{M - m_1}\right)\frac{V_0}{2}}$$

downward

(iii) $P_x = M\dot{x} = \text{constant} \times M$

$$\Rightarrow \quad M V_0 \cos\alpha = (M - m_1) V$$

and so,
$$\boxed{V_x = \frac{M}{M - m_1} V_0 \cos\alpha}$$

$$\cancel{\Rightarrow \frac{m_1}{M - m_1} \frac{V_0}{2} =}$$

at "burst"

(iv) Energy; $\frac{1}{2} M V_0^2 = \frac{1}{2} m_1 \frac{V_0^2}{4} + \frac{1}{2}(M - m_1)\left(V_z^2 + V_x^2\right)$

$$\frac{1}{2}\cancel{V_0^2}\left(M - \frac{m_1}{4}\right) = \frac{M - m_1}{2}\left(\left[\frac{m_1}{M - m_1}\right]^2 \frac{V_0^2}{4} + \left[\frac{M}{M - m_1}\right]^2 V_0^2 \cos^2\alpha\right)$$

$$= \frac{1}{2}\cancel{V_0^2}\left(\frac{1}{4}\frac{m_1^2}{(M - m_1)} + \frac{M^2}{(M - m_1)}\cos^2\alpha\right)$$

$$\left(M - \frac{m_1}{4}\right)(M - m_1) = \frac{1}{4} m_1^2 + M^2 \cos^2\alpha$$

$$M^2 - \frac{5}{4} m_1 M + \cancel{\frac{1}{4} m_1^2} = \cancel{\frac{1}{4} m_1^2} + M^2 \cos^2\alpha$$

$$m_1 = \frac{4}{5}\left(M - M\cos^2\alpha\right)$$

$$\boxed{m_1 = \frac{4}{5} M \sin^2\alpha}$$

hmmm, explosion adds energy to particles. If explosion is isotropic
$\Rightarrow$ no momentum is added because we regard $M_1 = M_2$

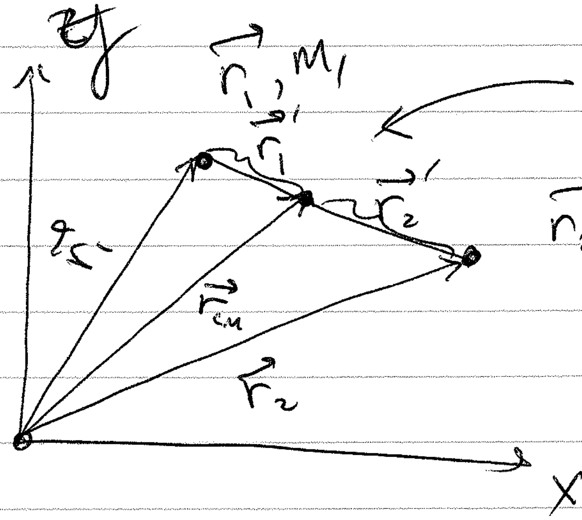
7.11

Show that the angular momentum of a two particle system is

$$\vec{r}_{CM} \times M \vec{V}_{cm} + \vec{R} \times \mu \vec{v} \leftarrow \text{relative velocity}$$

$M = m_1 + m_2$ Relative position vector   reduced mass

Sol^n



$$\vec{R} = \vec{r}_2 - \vec{r}_1 = \vec{r}_2{}' - \vec{r}_1{}'$$

$\vec{r}_2, m_2 \Rightarrow \dot{\vec{R}} = \dot{\vec{r}}_2{}' - \dot{\vec{r}}_1{}'$

(i) $\vec{L} = m_1 \vec{r}_1 \times \vec{V}_1 + m_2 \vec{r}_2 \times \vec{V}_2$

note: $\vec{r}_1 = \vec{r}_{cm} + \vec{r}_1{}'$ & $\vec{r}_2 = \vec{r}_{CM} + \vec{r}_2{}'$

$\dot{\vec{r}}_1 = \dot{\vec{r}}_{cm} + \dot{\vec{r}}_1{}'$ $\dot{\vec{r}}_2 = \dot{\vec{r}}_{cm} + \dot{\vec{r}}_2{}'$

$$= m_1\left[(\vec{r}_{cm} + \vec{r}_1{}') \times (\dot{\vec{r}}_{cm} + \dot{\vec{r}}_1{}')\right] + m_2\left[(\vec{r}_{CM} + \vec{r}_2{}') \times (\dot{\vec{r}}_{cm} + \dot{\vec{r}}_2{}')\right]$$

$$= (m_1 + m_2)(\vec{r}_{cm} \times \dot{\vec{r}}_{cm}) + m_1(\vec{r}_1{}' \times \dot{\vec{r}}_1{}') + m_2(\vec{r}_2{}' \times \dot{\vec{r}}_2{}')$$

$$+ \vec{r}_{cm} \times (\underbrace{m_1 \dot{\vec{r}}_1{}' + m_2 \dot{\vec{r}}_2{}'}_{\to 0}) + (\underbrace{m_1 \vec{r}_1{}' + m_2 \vec{r}_2{}'}_{\to 0}) \times \dot{\vec{r}}_{cm}$$

$$= m\left(\vec{r}_{cm} \times \dot{\vec{r}}_{cm}\right) + m_1\left(\frac{\vec{R} \times \dot{\vec{R}}}{\left(1+\frac{m_1}{m_2}\right)^2}\right) + m_2\left(\frac{\vec{R} \times \dot{\vec{R}}}{\left(1+\frac{m_2}{m_1}\right)^2}\right)$$

$$= m\left(\vec{r}_{cm} \times \dot{\vec{r}}_{cm}\right) + \left(\vec{R} \times \dot{\vec{R}}\right)\left[\frac{m_1 m_2^2}{(m_1+m_2)^2} + \frac{m_1^2 m_2}{(m_1+m_2)^2}\right]$$

$$= m\left(\vec{r}_{cm} \times \dot{\vec{r}}_{cm}\right) + \left(\vec{R} \times \dot{\vec{R}}\right)\underbrace{\left(\frac{m_1 m_2}{(m_1+m_2)^2}\right)\left(m_1+m_2\right)}_{\mu}$$

$$\boxed{\vec{L} = m\left(\vec{r} \times \dot{\vec{r}}_{cm}\right) + \vec{R} \times \mu \dot{\vec{R}}}$$

7.12

Cyg X-1 ; $P_{orb} = 5.6$ days

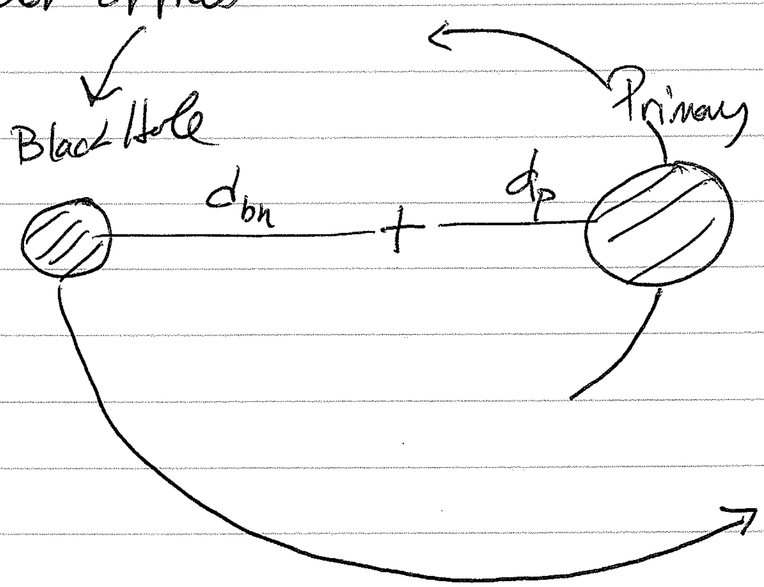
$M_p = 20\ M_\odot$

$M_{bh} = 16\ M_\odot$

find the semi-major axis of the Cyg X-1 system

Solution. Find Kepler's 3rd law of Motion.

Assume circular orbits



a) Primary: $$M_p \ddot{r}_p = -\frac{G M_{bh} M_p}{(d_{bh}+d_p)^2} + M_p \left(\frac{2\pi d_p}{P}\right)^2 \frac{1}{d_p}$$

Black Hole: $$M_{bh} \ddot{r}_{bh} = -\frac{G M_p M_{bh}}{(d_{bh}+d_p)^2} + M_{bh} \left(\frac{2\pi d_{bh}}{P}\right)^2 \frac{1}{d_{bh}}$$

semi-major axis $a = \cancel{\tfrac{1}{2}}(d_{bh} + d_p)$

b) from center-of-mass $\Rightarrow$ $M_p d_p = M_{bh} d_{bh}$

$$\frac{M_{bh}}{M_p} d_{bh} = d_p$$

$$\text{or } d_p = \left(\frac{M_{bh}}{M_p}\right)\underbrace{\left(a - d_p\right)}_{a \text{ definition}}$$

$$\text{or } d_p = \frac{a\,(M_{bh}/M_p)}{1 + (M_{bh}/M_p)}$$

$$d_p = a\left[\frac{M_{bh}}{M_p + M_{bh}}\right]$$

c) from Primary E-of-Motion,

$$-\frac{G\cancel{M_{bh}}\cancel{M_p}}{a^2} + \frac{4\pi^2 \cancel{M_p}}{P^2}\, a\left[\frac{\cancel{M_{bh}}}{M_p + M_{bh}}\right] = 0$$

$$a^3 = \frac{G(M_p + M_{bh})}{4\pi^2} P^2$$

$$\boxed{a = \left[\frac{G(M_p + M_{bh})}{4\pi^2}\right]^{\frac{1}{3}} P^{2/3}}$$

$\simeq 0.206$ A.U.

for $M_p = 20 M_\odot$, $M_{bh} = 16 M_\odot$,

$P = 5.6$ days

7.13

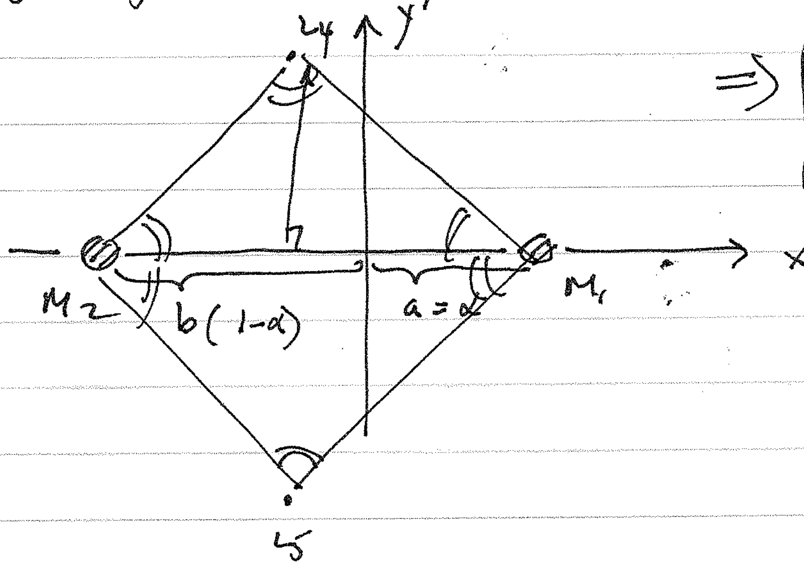
a) Using the coordinate convention given in Section 7.4 for the restricted 3-body problem, find the (x', y')-coordinates of the two Lagrangian points L_4 and L_5.

(i) $$V(x',y') = -\frac{1-\alpha}{\sqrt{(x'-\alpha)^2+y'^2}} - \frac{\alpha}{\sqrt{(x'+1-\alpha)^2+y'^2}} - \frac{x'^2+y'^2}{2}$$

$$\vec{\nabla} V = \left[\frac{(1-\alpha)(x'-\alpha)}{([x'-\alpha]^2+y'^2)^{3/2}}\hat{x}' + \frac{\alpha(x'+1-\alpha)}{([x'+1-\alpha]^2+y'^2)^{3/2}}\hat{x}' - x'\hat{x}'\right]$$

$$+ \left[\frac{(1-\alpha)\,y'}{([x'-\alpha]^2+y'^2)^{3/2}}\hat{y}' + \frac{\alpha y'}{([x'+1-\alpha]^2+y'^2)^{3/2}}\hat{y}' - y'\hat{y}'\right]$$

L points fall where $\vec{\nabla} V = 0$. Show that this is true.



$\Rightarrow$ L_4, L_5 sit at
$$\begin{cases} x' = (\alpha - 0.5) \\ y' = \pm 0.866 = \pm\sqrt{3}/2 \end{cases}$$

(ii) $L_4, L_5 \longrightarrow \vec{\nabla} V = 0\hat{x}' + 0\hat{y}'$