Homeowrk 2
Due: 2012 October 12

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13. Two uniformly charged spheres, each of total charge Q and radius R, are separated by distance $r > 2R$. Show that the force between the two spheres is

$$\mathbf{F} = \frac{1}{4\pi\epsilon_o}\frac{QQ}{r^2}\hat{\mathcal{R}} \tag{1}$$

where $\mathcal{R}$ is the vector pointing from the source charge to the field charge.
14. The Yukawa potential is given by

$$V = \kappa\,\frac{e^{-\alpha r}}{r} \tag{2}$$

where κ and α are constants.

a. Find the electric field associated with the Yukawa potential.

b. Find the charge distribution that leads to the Yukawa potential.

c. Find the total charge of the system.

⑧ Problem 2.12

Use Gauss's law to find $\vec{E}$ inside a uniformly charged sphere, charge density $\rho = Q/(4\pi R^3/3)$ where R is the radius of the sphere and Q is the total charge of the sphere.

Sol'n

$$\oint \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int \rho \, d^3x$$

By symmetry (charge is distributed isotropically), $\vec{E}$ is radial

$$\Rightarrow E_r 4\pi r^2 = \frac{1}{\epsilon_0}\int_0^r \rho \left[r'^2 dr' \underbrace{\sin\theta' d\theta' d\phi'}_{\text{integrate over } \theta', \phi'}\right]$$

$$\Rightarrow \int d\theta' \sin\theta' d\phi' = 4\pi$$

$$= \frac{1}{\epsilon_0} 4\pi\rho \int_0^r r'^2 dr'$$

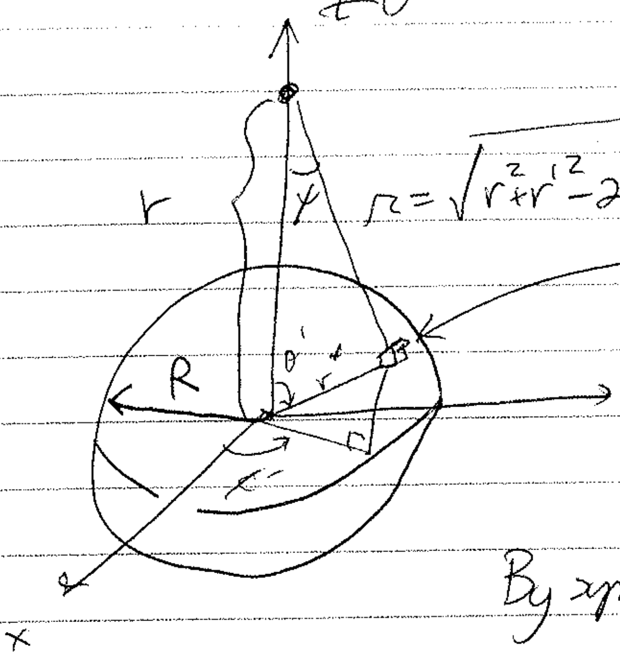
$$E_r 4\pi r^2 = \frac{1}{\epsilon_0}\frac{4\pi\rho}{3} r^3 \Rightarrow \boxed{E_r = \frac{\rho}{3\epsilon_0} r, \quad r < R}$$

This is the same as 2-8 (found by direct integration of

$$\vec{E}_r = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')\, d^3x' \, \hat{\imath}}{\imath^2}$$

(see next 2 pages)

Use a direct integration:



$\mathscr{r}=\sqrt{r^2+r'^2-2rr'\cos\theta'}$

$dq=\rho\underbrace{r'^2dr'\sin\theta'd\theta'd\phi'}_{d^3x'}$

$\Rightarrow d\vec{E}=\frac{1}{4\pi\epsilon_0}\frac{dq\,\hat{\mathscr{r}}}{\mathscr{r}^2}$

By symmetry we need only find E_z

$\Rightarrow dE_z=\hat{z}\cdot d\vec{E}=\frac{1}{4\pi\epsilon_0}\frac{dq}{\mathscr{r}^2}\underbrace{\left[\frac{r-r'\cos\theta'}{\mathscr{r}}\right]}_{\cos\psi}$

$$\Rightarrow E_z=\frac{1}{4\pi\epsilon_0}\int\frac{\rho r'^2dr'\sin\theta'd\theta'd\phi'(r-r'\cos\theta')}{(r^2+r'^2-2rr'\cos\theta')^{3/2}}$$

① the $\int d\phi'$ integration yields 2π

② let's do $d\theta'$ integration next (but only consider $r<R$)

$$=\frac{\rho}{2\epsilon_0}\int_0^R r'^2dr'\int_0^\pi \sin\theta'd\theta'\left(\frac{(r-r'\cos\theta')}{(r^2+r'^2-2rr'\cos\theta')^{3/2}}\right)$$

let $W=r^2+r'^2-2rr'\cos\theta' \Rightarrow dW=+2rr'\sin\theta'd\theta'$

$$=\frac{\rho}{2\epsilon_0}\int_0^R r'^2dr'\int_{(r-r')^2}^{(r+r')^2}\left(+\frac{dW}{2rr'}\right)\left[\frac{\frac{W-r^2-r'^2}{2r}+r}{W^{3/2}}\right]$$

$$= +\frac{\rho}{4\epsilon_0 r}\int_0^R r'dr' \int^{(r+r')^2} \left[\frac{r^2 - \frac{1}{2}(r^2+r'^2)}{rW^{3/2}} + \frac{1}{2rW^{1/2}}\right] dW$$

$$= +\frac{\rho}{4\epsilon_0 r^2}\int_0^R r'dr' \int_{(r-r')^2}^{(r+r')^2} \left[\frac{\frac{1}{2}(r^2-r'^2)}{W^{3/2}} + \frac{1}{2W^{1/2}}\right] dW$$

$$= +\frac{\rho}{4\epsilon_0 r^2}\int_0^R r'dr' \left[\frac{\frac{1}{2}(r^2-r'^2)}{W^{1/2}}(-2) + \frac{2W^{1/2}}{2}\right]_{(r-r')^2}^{(r+r')^2}$$

we need to break up the r' integral into 2 parts, $r' > r$ and $r' < r$

$$= +\frac{\rho}{4\epsilon_0 r^2}\left\{\int_0^r r'dr'\left[-(r^2-r'^2)\underbrace{\left(\frac{1}{r+r'} - \frac{1}{r-r'}\right)}_{\frac{-2r'}{r^2-r'^2}} + \underbrace{(r+r') - (r-r')}_{2r'}\right]\right.$$

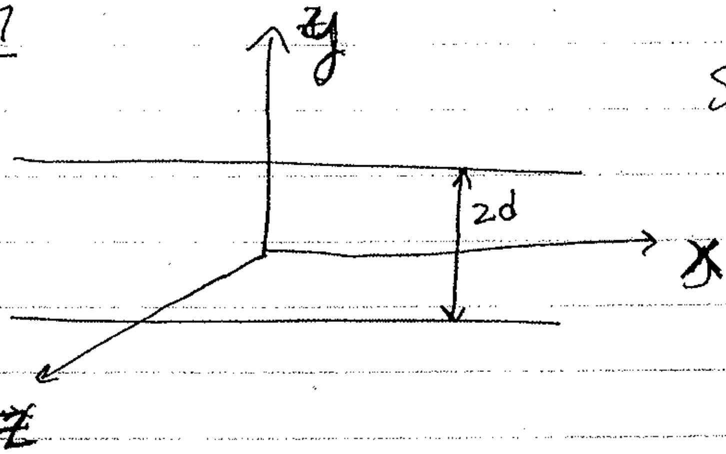
$$\left. + \int_r^R r'dr'\left[-(r^2-r'^2)\underbrace{\left(\frac{1}{r+r'} - \frac{1}{r'-r}\right)}_{\frac{2r}{r^2-r'^2}} + \underbrace{(r+r') - (r'-r)}_{2r}\right]\right\}$$

the $\int_r^R r'dr'$ integral $\rightarrow 0$, $r' > R$ makes no contribution (as we already know)!

$$= +\frac{\rho}{4\epsilon_0 r^2}\int_0^r r'dr'\left[+2r' + 2r'\right]$$

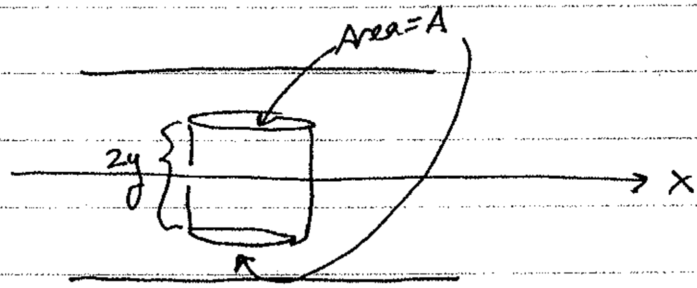
$$= +\frac{\rho}{4\epsilon_0 r^2}\left[\frac{4}{3}r'^3\Big|_0^r\right] = +\frac{\rho r}{3\epsilon_0}$$

(9) 2.17



Slab has uniform charge density ρ. We see from symmetry arguments that there will only be a y-component for the field. Let's Use Gauss's law to find E_y.

a) $|y| < d$ ← inside the slab



$$\oint \vec{E} \cdot d\vec{S} = \int \frac{\rho \, d\tau}{\epsilon_0}$$

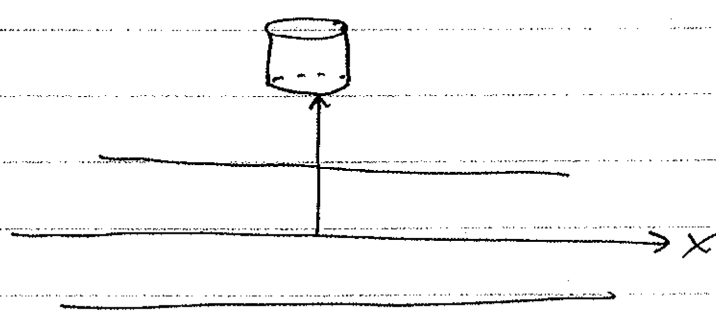
$$E_y^+ A - E_y^- A = 2Ay \frac{\rho}{\epsilon_0}$$

are in opposite directions

$$2E_y = 2y\rho/\epsilon_0$$

$$\rightarrow \boxed{\vec{E}_y = \frac{\rho y}{\epsilon_0} \hat{y}}$$

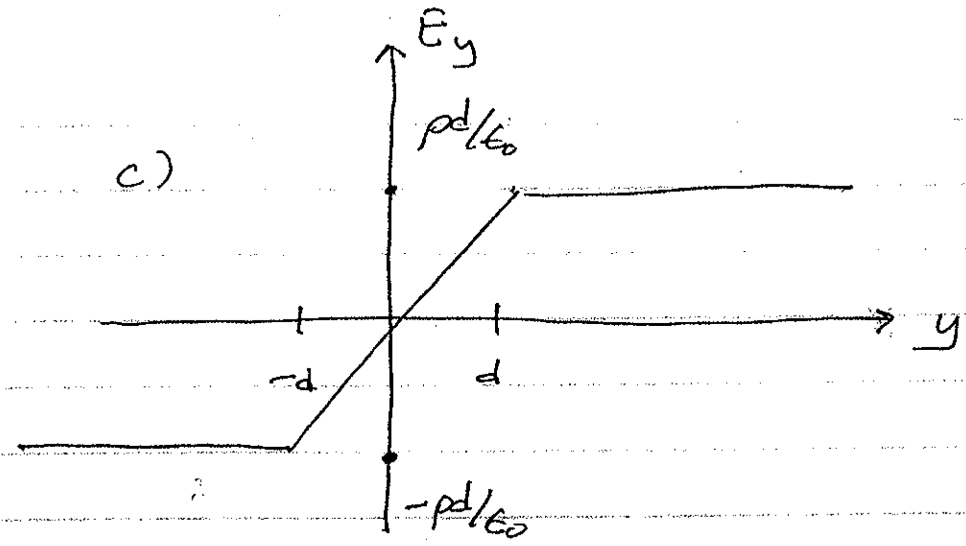
b) Outside slab



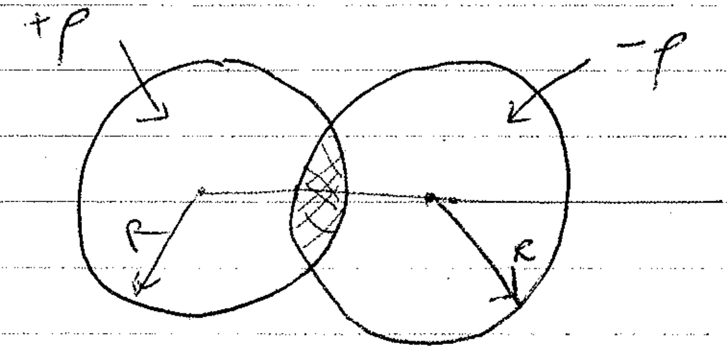
$$E_y^+ A - E_y^- A = 0$$

$$\rightarrow E_y^+ = E_y^-$$

E_y is constant above & below the slab c)



(10) Prob 2.18



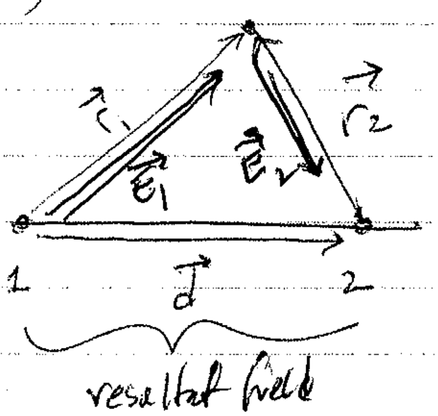
find $\vec{E}$ in the overlap region

(a) the $\vec{E}$ of a uniformly charged sphere is

$$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int \rho \, d^3x$$

$$E_r \, 4\pi r^2 = \frac{1}{\epsilon_0} 4\pi \rho_0 \frac{r^3}{3} \longrightarrow \boxed{\vec{E}_r = \frac{\rho_0}{3\epsilon_0} r \hat{r}}$$

(b) thus



and $\vec{E}_{Total} = \frac{\rho_0}{3\epsilon_0} \vec{r}_1 - \frac{\rho_0}{3\epsilon_0} \vec{r}_2$

$$= \frac{\rho}{3\epsilon_0} (\vec{r}_1 - \vec{r}_2)$$

$$\boxed{\vec{E}_{Tot} = \frac{\rho}{3\epsilon_0} \vec{d}}$$

(11) 2.19 Calculate $\vec{\nabla}\times\vec{E}$ from $\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')d\tau'}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$

$$\Rightarrow \vec{\nabla}_r \times \vec{E} = \frac{1}{4\pi\epsilon_0}\vec{\nabla}_r \times \int \rho(\vec{r}')d\tau' \left[\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right]$$

to keep it separate from $\vec{r}'$, the integration variable

a) does not operate on $\vec{r}'$ $\Rightarrow$ can be dragged into the integral and moved past $\rho(\vec{r}')\,d\tau'$, if desired. Can do this and find the answer

b) however, let's do it another way. look at $\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$. Note that

$$\vec{\nabla}_r\left[\frac{1}{|\vec{r}-\vec{r}'|}\right] = \vec{\nabla}_r\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right]$$

$$= \hat{x}\left[\frac{-(x-x')}{|\vec{r}-\vec{r}'|^3}\right] + \hat{y}\left[\frac{-(y-y')}{|\vec{r}-\vec{r}'|^3}\right] + \hat{z}\left[\frac{-(z-z')}{|\vec{r}-\vec{r}'|^3}\right]$$

$$= -\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$\Rightarrow \vec{\nabla}_r\times\vec{E} = \frac{1}{4\pi\epsilon_0}\vec{\nabla}_r\times\int\rho(\vec{r}')d\tau'\left[-\vec{\nabla}_r\frac{1}{|\vec{r}-\vec{r}'|}\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\vec{\nabla}_r\times\vec{\nabla}_r\underbrace{\int\left\{-\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\right\}d\tau'}_{\text{scalar function}}$$

$$\Rightarrow \boxed{\vec{\nabla}_r\times\vec{E} = 0}$$

(12) 2.20

(I) which field is electrostatic?

a) $\vec{E} = k\left[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z} \right]$

$\vec{\nabla}\times\vec{E} = k(0-2y, 0-3z, 0-x) \neq 0$

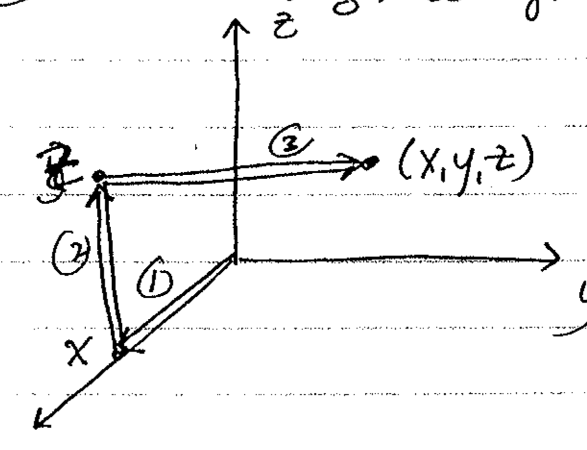
→ not electrostatic

b) $\vec{E} = k(y^2\hat{x} + [2xy + z^2]\hat{y} + 2yz\hat{z})$

$\vec{\nabla}\times\vec{E} = k(2z-2z, 0-0, 2y-2y) = 0$

→ electrostatic

(II) Find V using the origin as the reference point



Take the 3 paths as shown to the left:

① $x' = [0, x]$, $y' = 0$, $z' = 0$

② $x' = x$, $y' = 0$, $z' = [0, z]$

③ $x' = x$, $y' = [0, y]$, $z' = z$

$$-\int dV = \int_0^x k\cancelto{0}{y^2}dx + \int_0^z 2k\cancelto{0}{y}z\,dz + \int_0^y k(2xy + z^2)dy = (xy^2 + yz^2)k$$

$$\Rightarrow -V(x,y,z) + V(0,0,0) = (xy^2 + yz^2)k$$

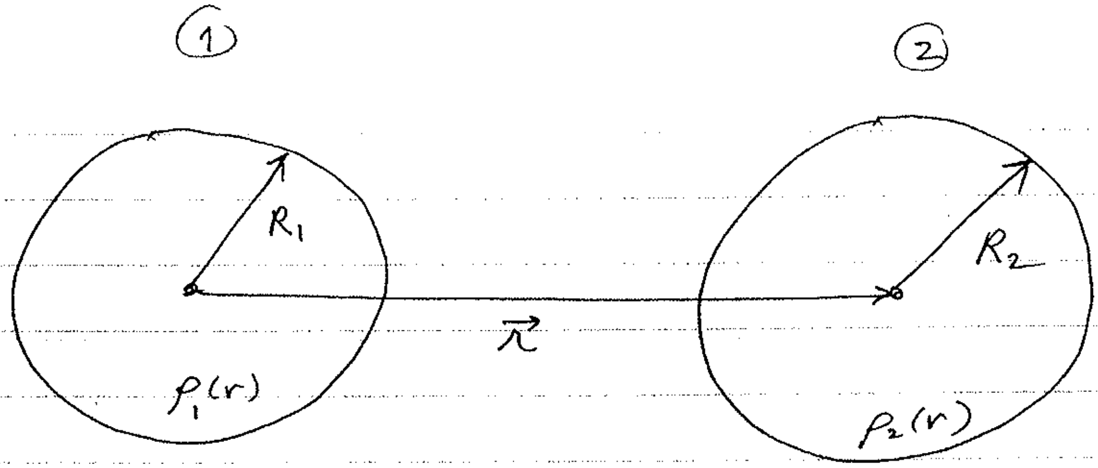
$$\boxed{V(x,y,z) = \underbrace{V(0,0,0)}_{\text{set to } 0} - k(xy^2 + yz^2)}$$

(IV) Find $\vec{E} = -\vec{\nabla} V$

$$= k\left[-y^2\hat{x} - (2xy + z^2)\hat{y} - 2zy\,\hat{z}\right]$$

$$\boxed{\vec{E} = k\left(y^2\hat{x} + (2xy + z^2)\hat{y} + 2zy\,\hat{z}\right)}$$

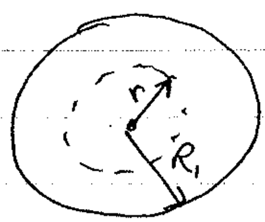
(B)



Find force of sphere 1 on sphere 2

(a) Find field of sphere 1. Use Gauss's law because of symmetry

$$\oint \underbrace{\vec{E}\cdot d\vec{S}}_{\text{on sphere w/radius } r} = \underbrace{\frac{\int \rho\, d^3x}{\epsilon_0}}_{\text{over sphere w/radius } r}$$



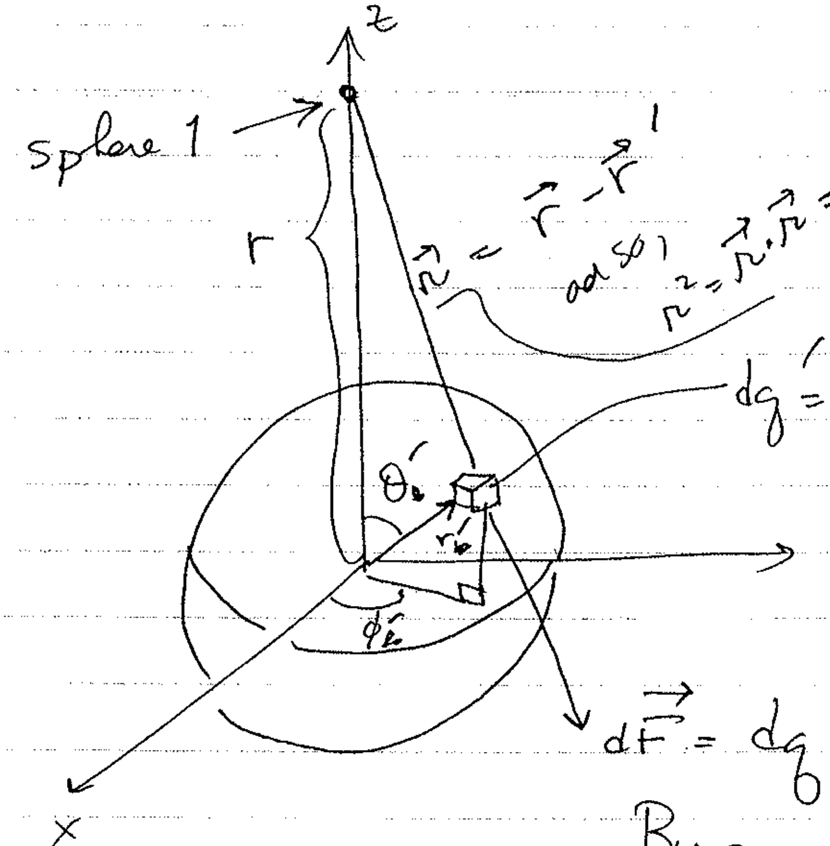
$$E_r 4\pi r^2 = \frac{Q(r)}{\epsilon_0}$$

charge contained in sphere of radius r

and for $r > R_1$ $\Longrightarrow$ $\boxed{E_r = \dfrac{Q_1}{4\pi\epsilon_0 r^2}}$

the field acts as if all of the charge were concentrated at the center of the sphere.

(b) Find force of sphere 2



$$dq' = \rho_2 r'^2 \sin\theta' d\theta' d\phi' dr'$$

$$d\vec{F} = dq'\vec{E}_1 \equiv \text{force exerted on } dq'$$

By symmetry, we need only consider the force along z.

$$\Longrightarrow dF_z = \hat{z}'\cdot d\vec{F}' = (dq'E_1)\left[\frac{r - r'\cos\theta'}{|\vec{\mathscr{r}}|}\right]$$

$$F_z = \int \rho_2 r'^2 dr' \sin\theta' d\theta' d\phi' \left[\frac{Q_1}{4\pi\epsilon_0}\left(\frac{r - r'\cos\theta'}{\{r^2 + r'^2 - 2rr'\cos\theta'\}^{3/2}}\right)\right]$$

$$\left\{ \begin{array}{l} \underline{\text{note}}: \int d\phi' = 2\pi \\ \underline{\text{note}}\ \ \sin\theta d\theta = d(-\cos\theta) \end{array} \right.$$

$$\Longrightarrow F_z = \frac{-Q_1}{2\epsilon_0}\int \rho_2 r'^2 dr' \left[\frac{r - r'\cos\theta'}{(r^2 + r'^2 - 2rr'\cos\theta')^{3/2}}\right] d(\cos\theta')$$

let: $W = r^2 + r'^2 - 2rr'\cos\theta'$

$$dW = -2rr'\,d(\cos\theta')$$

$$\rightarrow = \frac{Q_1}{2\epsilon_0}\int \rho_2 r'^2 dr' \left[\frac{r - r'\left(\frac{r^2+r'^2-W}{2rr'}\right)}{W^{3/2}}\right]\frac{dW}{2rr'}$$

$$= \frac{Q_1}{4\epsilon_0 r}\int \rho_2 r' dr' \left[\frac{\overbrace{r - \frac{1}{2}\frac{r^2+r'^2}{r}}^{\frac{1}{2r}(r^2-r'^2)}}{W^{3/2}} + \frac{1}{2r}\frac{1}{W^{1/2}}\right] dW$$

$$= \frac{Q_1}{4\epsilon_0 r}\int \rho_2 r' dr' \left[\frac{(r^2-r'^2)}{\cancel{2}r}\frac{-\cancel{2}}{W^{1/2}} + \frac{1}{\cancel{2}r}\cancel{2}W^{1/2}\right]_{w_0=(r-r')^2}^{w_1=(r+r')^2}$$

$$= \frac{Q_1}{4\epsilon_0 r^2}\int \rho_2 r' dr' \left[-(r^2-r'^2)\left[\frac{1}{(r+r')} - \frac{1}{(r-r')}\right] + (r+r') - (r-r')\right]$$

$$= \frac{Q}{4\epsilon_0 r^2}\int \rho_2 r' dr' \left[\underbrace{-(r-r') + (r+r') + 2r'}_{4r'}\right]$$

$$= \frac{Q_1}{4\epsilon_0 r^2}\int 4\rho_2 r'^2 dr'$$

In spherical symmetry, this integral is

$$= \frac{Q_1}{\epsilon_0 r^2}\underbrace{\int \rho_2 r'^2 dr'}_{\frac{Q_2}{4\pi}}$$

$$\rightarrow \boxed{F_z = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^2}}$$

; behaves as if both spheres were point charges!

(14) Yukaw Pot^l (Problem 2.46 in text)

$$V = K\frac{e^{-r/r_0}}{r}$$

$$\rightarrow \vec{E} = -\vec{\nabla} V = -K\hat{r}\left[-\frac{e^{-r/r_0}}{rr_0} - \frac{e^{-r/r_0}}{r^2}\right]$$

$$\boxed{\vec{E} = Ke^{-r/r_0}\left[\frac{\hat{r}}{rr_0} + \frac{\hat{r}}{r^2}\right]}$$

a) find $\rho(r)$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\rightarrow \rho/\epsilon_0 = \left[K\frac{e^{-r/r_0}}{r_0}\left(\vec{\nabla}\cdot\frac{\hat{r}}{r}\right) + \frac{\hat{r}}{r}\cdot\vec{\nabla}K\frac{e^{-r/r_0}}{r_0}\right.$$
$$\left. + Ke^{-r/r_0}\left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2}\cdot\vec{\nabla}Ke^{-r/r_0}\right]$$

$$= K\frac{e^{-r/r_0}}{r_0}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2/r\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(K\frac{e^{-r/r_0}}{r_0}\right)$$
$$+ Ke^{-r/r_0}4\pi\delta^3(\vec{r}) + \frac{1}{r^2}\left(\frac{\partial}{\partial r}Ke^{-r/r_0}\right)$$

$$= \frac{Ke^{-r/r_0}}{r_0}\left(\frac{1}{r^2} - \frac{1}{rr_0} + 4\pi\delta^3(\vec{r})r_0 - \frac{1}{r^2}\right)$$

$$\boxed{\frac{\rho}{\epsilon_0} = K\frac{e^{-r/r_0}}{r_0}\left(r_0 4\pi\delta^3(\vec{r}) - \frac{1}{rr_0}\right)}$$

b) $Q = \int_0^\infty \rho \, d^3x$

$= \frac{K\epsilon_0}{r_0} \int_0^\infty \left(4\pi \delta^3(\vec{r}) r_0 - \frac{1}{r r_0} \right) e^{-r/r_0} d^3x$

$= \left[\frac{4\pi K r_0}{r_0} - \frac{K}{r_0} \int_0^\infty \frac{e^{-r/r_0}}{r r_0} 4\pi r^2 dr \right] \epsilon_0$

$= \frac{4\pi K\epsilon_0}{r_0} \left[r_0 - \frac{1}{r_0} \int_0^\infty \underbrace{r e^{-r/r_0} dr}_{\text{by parts}} \right]$

let: $U = r$, $dV = e^{-r/r_0} dr$

$dU = dr \quad V = -r_0 e^{-r/r_0}$

$\rightarrow Q = \frac{4\pi}{r_0} K\epsilon_0 \left[r_0 - \frac{1}{r_0} \left\{ \cancel{-r r_0 e^{-\frac{r}{r_0}} \Big|_0^\infty}^{\,0} + r_0 \int_0^\infty e^{-\frac{r}{r_0}} dr \right\} \right]$

$= \frac{4\pi\epsilon_0}{r_0} K \left[r_0 - \frac{1}{r_0} \left\{ r_0 \times (-r_0) e^{-r/r_0} \Big|_0^\infty \right\} \right]$

$= \frac{4\pi\epsilon_0}{r_0} K \left[r_0 + r_0 (0 - 1) \right]$

$= 0$!

the total charge of the configuration which leads to $V = K\frac{e^{-r/r_0}}{r}$ is 0; the object is neutral