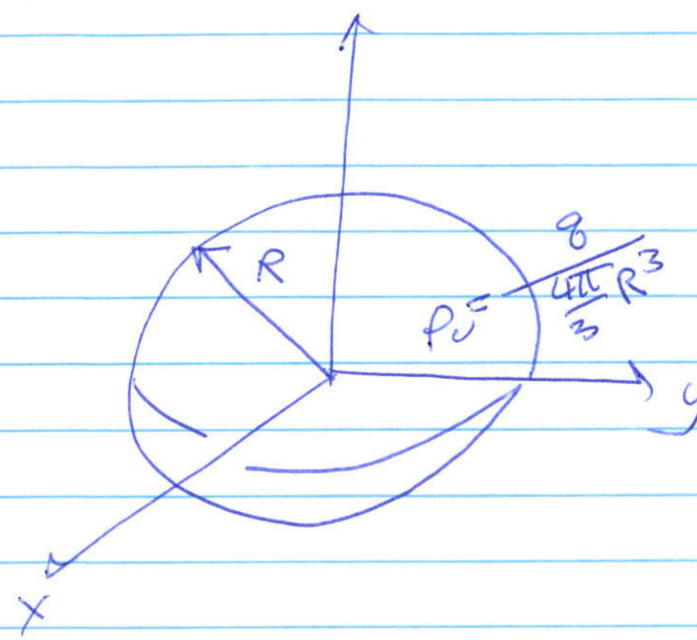
Homeowrk 3
Due: 2012 October 19

15. Problem 2.21
16. Problem 2.26
17. Problem 2.28
18. Problem 2.32
19. The Yukawa potential is given by

$$V = \kappa \frac{e^{-\alpha r}}{r} \tag{1}$$

where κ and α are constants. Find the energy for the charge distribution responsible for the Yukawa potential.
20. Problem 2.47
21. Problem 2.48

15 Prob 2.21



Find $V(r)$ for an uniformly charged sphere of radius R and total charge q

(i) Find $\vec{E}$ using Gauss's law and then find $V(r)$ from

$$-dV = \vec{E} \cdot d\vec{r}$$

(i) $Q(r) = \int \rho_0 d^3x$

$$= \begin{cases} \rho_0 \frac{4\pi}{3} r^3 = q\frac{r^3}{R^3}, & r < R \\ q, & r > R \end{cases}$$

$$\Rightarrow \vec{E}(r) = \frac{q\hat{r}}{4\pi\epsilon_0} \begin{cases} \frac{r}{R^3}, & r < R \\ \frac{1}{r^2}, & r > R \end{cases}$$

$$-\int_{\infty}^{r} dV = \int_{\infty}^{r} \frac{q\hat{r} \cdot d\vec{r}}{4\pi\epsilon_0 r^2}, \quad r > R$$

$$-V(r) + \cancel{V(\infty)}^{\,0} = \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r}\Big|_{\infty}^{r} \right]$$

$$\boxed{V(r) = \frac{q}{4\pi\epsilon_0 r}\quad , r > R}$$

Now do $r < R$

$$-\int_{\infty}^{r} dV = \int_{\infty}^{R} \frac{q\hat{r}\cdot d\vec{r}}{4\pi\epsilon_0 r^2} + \int_{R}^{r} \frac{q\, r\hat{r}\cdot d\vec{r}}{4\pi\epsilon_0 R^3}$$

$$-V(r) + \cancel{V(\infty)}^{\,0} = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)\Big|_{\infty}^{R} + \frac{q}{4\pi\epsilon_0 R^3}\left(\frac{r^2}{2}\right)\Big|_{R}^{r}$$

$$-V(r) = -\frac{q}{4\pi\epsilon_0 R} + \frac{q}{8\pi\epsilon_0 R^3}\left(r^2 - R^2\right)$$

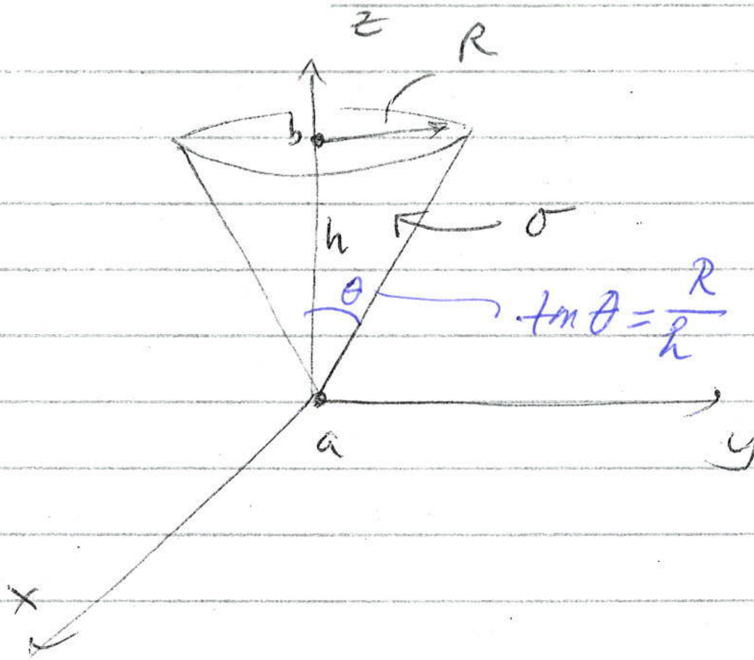
$$V(r) = \frac{q}{4\pi\epsilon_0 R}\left[1 - \frac{1}{2}\left(\frac{r^2}{R^2} - 1\right)\right]$$

$$\boxed{V(r) = \frac{q}{4\pi\epsilon_0 R}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right)\quad , r < R}$$

find $\vec{E} = -\vec{\nabla} V$

$$\vec{E} = \frac{q\hat{r}}{4\pi\epsilon_0}\begin{cases} \frac{1}{r^2} & r > R \\ +\frac{r}{R^3} & r < R \end{cases}$$

(16) Prob 2.26

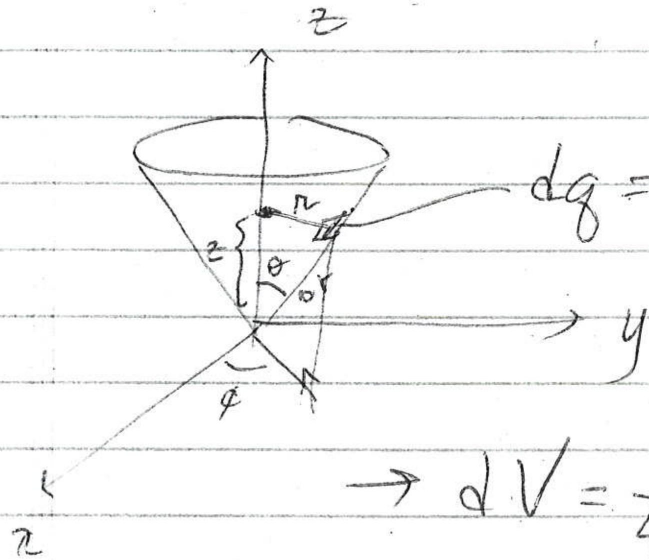


$\tan\theta = \frac{R}{h}$

Find ΔV between b and a

Sol$^{\underline{n}}$

Find V on the z-axis; Use spherical coordinates



$dq = \sigma\, r\, d\phi\, dr \sin\theta$

where $\sin\theta = \frac{R}{\sqrt{h^2+R^2}}$

$$\rightarrow dV = \frac{1}{4\pi\epsilon_0} \frac{\sigma r\, d\phi\, dr \sin\theta}{\sqrt{r^2+z^2-2zr\cos\theta}}$$

$$V = \frac{\sigma}{4\pi\epsilon_0}\int \frac{r\,dr\,d\phi \sin\theta}{\sqrt{r^2+z^2-2zr\cos\theta}}$$

Integrate $\int d\phi$

$$V = \frac{\sigma}{2\epsilon_0}\int_0^{r_0} \frac{r\,dr\,\sin\theta}{\sqrt{r^2+z^2-2zr\cos\theta}}$$

$$= \frac{\sigma\sin\theta}{2\epsilon_0}\Big[\sqrt{r^2-2zr\cos\theta+z^2} + z\cos\theta\ln\left\{-z\cos\theta+r+\sqrt{r^2-2zr\cos\theta+z^2}\right\}\Big]_0^{r_0}$$

okay; integration limits are 0 and r_0, where

$$r_0 = \sqrt{h^2+R^2}$$

okay; what are $\cos\theta$, ... ?

$$\cos\theta = \frac{h}{\sqrt{h^2+R^2}} = \frac{h}{r_0} \Rightarrow r_0 = \frac{h}{\cos\theta}$$

Evaluate integral

$$V = \frac{\sigma\sin\theta}{2\epsilon_0}\Bigg[\sqrt{h^2+R^2-2zh+z^2} + \frac{zh}{\sqrt{h^2+R^2}}\ln\left\{-\frac{zh}{\sqrt{h^2+R^2}}+\sqrt{h^2+R^2}+\sqrt{h^2+R^2-2zh+z^2}\right\}$$
$$-\left\{z+\frac{zh}{\sqrt{h^2+R^2}}\ln\left(-\frac{zh}{\sqrt{R^2+h^2}}+z\right)\right\}\Bigg]$$

find $\Delta V = V(h) - V(0)$

a) $V(0) = \frac{\sigma \sin\theta}{2\epsilon_0}\left[\left(\sqrt{h^2+R^2}+0\right)-0\right]$

$$\boxed{V(0) = \frac{\sigma R}{2\epsilon_0}}$$

b) $V(h) = \frac{\sigma \sin\theta}{2\epsilon_0}\left[R + \frac{h^2}{\sqrt{h^2+R^2}} \ln\left\{-\frac{h^2}{\sqrt{h^2+R^2}} + \sqrt{R^2+h^2}+R\right\}\right.$

$$\left. - \left\{h + \frac{h^2}{\sqrt{h^2+R^2}} \ln\left(h - \frac{h^2}{\sqrt{R^2+h^2}}\right)\right\}\right]$$

$$= \frac{\sigma \sin\theta}{2\epsilon_0}\left[(R-h) + \frac{h^2}{\sqrt{h^2+R^2}} \ln\left\{\frac{h^2+h^2+R^2+R\sqrt{h^2+R^2}}{h(R^2+h^2)^{1/2}-h^2}\right\}\right]$$

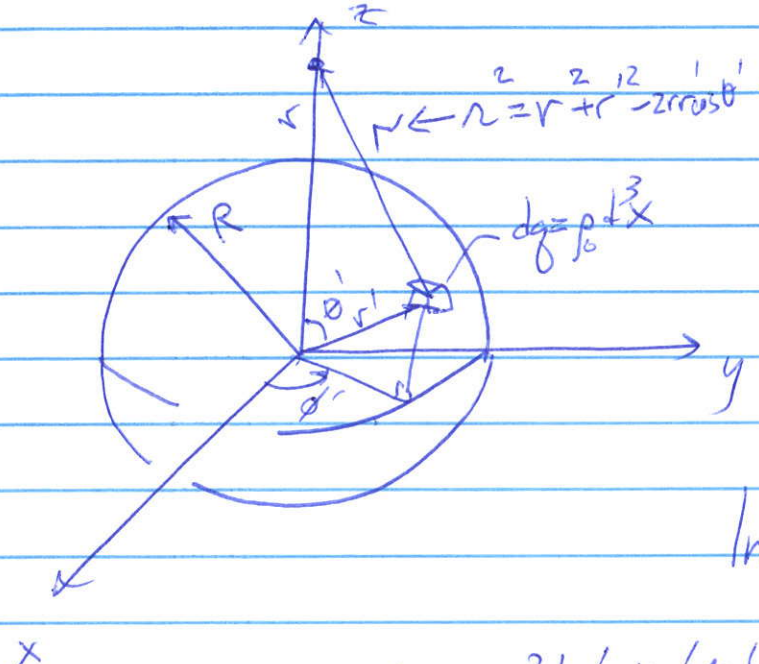
$$= \frac{\sigma \sin\theta}{2\epsilon_0}\left[(R-h) + \frac{h^2}{\sqrt{h^2+R^2}} \ln\left\{\frac{2h^2+R^2+R\sqrt{h^2+R^2}}{h\sqrt{R^2+h^2}-h^2}\right\}\right]$$

a) & b) give $\Delta V = V(h) - V(0)$

$$\Rightarrow \Delta V = \frac{\sigma \sin\theta}{2\epsilon_0}\left[(R-h) + \frac{h^2}{\sqrt{h^2+R^2}} \ln\left\{\frac{2h^2+R^2+R\sqrt{h^2+R^2}}{h\sqrt{R^2+h^2}-h^2}\right\} - \sqrt{h^2+R^2}\right]$$

in some books, $(R = h)$

(17) Prob 2.28



Radius R and total charge q for a uniformly charged sphere, the charge element $dq = \rho_0 d^3x$ produces a potential at r of

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{\mathcal{R}}$$

Integrate to find V

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_0 r'^2 dr' \sin\theta' d\theta' d\phi'}{\sqrt{r'^2 + r^2 - 2rr'\cos\theta'}}$$

Integrate over ϕ'

$$V = \frac{\rho_0}{2\epsilon_0}\int_0^R r'^2 dr' \int_0^\pi \sin\theta' d\theta' \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}}$$

let $W = r^2 + r'^2 - 2rr'\cos\theta' \Longrightarrow dW = 2rr'\sin\theta' d\theta'$

$$V = \frac{\rho_0}{2\epsilon_0}\int_0^R r'^2 dr' \int \frac{dW/2rr'}{W^{1/2}}$$

$$= \frac{\rho_0}{4\epsilon_0 r}\int_0^R r' dr' \left\{ 2W^{1/2} \Big|_{(r-r')^2}^{(r+r')^2} \right\}$$

Consider $r > R$ first

$$V = \frac{\rho_0}{2\epsilon_0 r}\int_0^R r' dr' \left[(r+r') - (r-r') \right]$$

$$V = \frac{\rho_0}{2\epsilon_0 r}\int_0^R r'dr'\left[2r'\right]$$

$$= \frac{\rho_0}{3\epsilon_0 r}\left[R^3\right]$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r}\right)}$$

Consider $r < R \Rightarrow$ we need to let the limits for W change if $r > r'$ or $r' > r$

$$V = \frac{\rho_0}{2\epsilon_0 r}\left[\int_0^r r'dr'\left[W^{1/2}\Big|_{(r-r')^2}^{(r+r')^2}\right] + \int_r^R r'dr'\left[W^{1/2}\Big|_{(r'-r)^2}^{(r+r')^2}\right]\right]$$

$r > r'$ $\qquad\qquad$ $r' > r$

$$= \frac{\rho_0}{2\epsilon_0 r}\left[\int_0^r r'dr'\left\{(r+r')-(r-r')\right\} + \int_r^R r'dr'\left\{(r+r')-(r'-r)\right\}\right]$$

$$= \frac{\rho_0}{2\epsilon_0 r}\left[\int_0^r r'dr'\left(2r'\right) + \int_r^R r'dr'\left(2r\right)\right]$$

$$= \frac{\rho_0}{\epsilon_0 r}\left[\left(\frac{r^3}{3} - 0\right) + \frac{r}{2}\left(R^2 - r^2\right)\right]$$

$$= \frac{\rho_0}{\epsilon_0}\left[\frac{r^2}{3} + \frac{R^2}{2} - \frac{r^2}{2}\right]$$

$$V(r) = \frac{\rho_0}{6\epsilon_0}\left[-r^2 + 3R^2\right]$$

$\rho_0 = 3q/4\pi R^3$

$$\Rightarrow V(r) = \frac{1}{6\epsilon_0}\left[\frac{3q}{4\pi R^3}\right] \times \left[3R^2 - r^2\right]$$

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{2}\frac{q}{R}\right)\left[1 - \frac{1}{3}\left(\frac{r^2}{R^2}\right)\right]}$$

find $\vec{E}(r) = -\vec{\nabla} V$

$$\vec{E}(r) = \begin{cases} +\dfrac{1}{4\pi\epsilon_0}\dfrac{q\hat{r}}{r^2}, & r > R \\ -\dfrac{1}{4\pi\epsilon_0}\left(\dfrac{\cancel{3}q}{\cancel{2}R}\right)\left(-\dfrac{\cancel{2}}{\cancel{3}}\dfrac{r}{R^2}\right)\hat{r}, & r < R \end{cases}$$

$$\vec{E}(r) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q\hat{r}}{r^2}, & r > R \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{q\hat{r}}{R^2}\left(\dfrac{r}{R}\right), & r < R \end{cases}$$

(18) Problem 2.32

Find the energy stored in a uniformly charged solid sphere of radius R and charge q. Do 3 different ways:

(a) Eq 2.43

(b) Eq 2.45

(c) Eq 2.44. Take a spherical shell of radius a. What happens as $a \to \infty$?

(a)

$$W = \frac{1}{2}\int \rho V \, d\tau \qquad (2.43)$$

(i) find V from $\rho = \begin{cases} \frac{3q}{4\pi R^3} & , r < R \\ 0 & , r > R \end{cases}$

from Gauss's law find $\vec{E}$. Need only the radial component from symmetry. For $r < R$

$$\Rightarrow E_r 4\pi r^2 = \frac{1}{\epsilon_0}\left[\frac{3q}{4\pi R^3}\right]\frac{4\pi}{3} r^3 = \frac{q}{\epsilon_0}\left(\frac{r}{R}\right)^3$$

$$\Rightarrow E_r = \frac{q}{4\pi\epsilon_0 R^3} r \quad , \quad r < R$$

and so, ~~$-dV = \vec{E}\cdot d\vec{r} = \frac{1}{4\pi\epsilon_0 R}$~~

For $r > R$

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \quad , r > R$$

$$\Rightarrow -\int_\infty^r dV = \int_\infty^r \vec{E}\cdot d\vec{r}$$

$$-V(r) + V_\infty = \int_\infty^R \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr + \int_R^r \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\,r\,dr$$

$$= -\frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \cancel{\frac{1}{\infty}}^{\,0}\right) + \frac{q}{4\pi\epsilon_0 R^3}\left(\frac{r^2}{2} - \frac{R^2}{2}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{r^2}{2R^3} - \frac{3}{2R}\right)$$

<u>set $V_\infty = 0$</u>

$$\Rightarrow V(r) = \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right), \quad r < R$$

find $$W = \frac{1}{2}\int_0^R \rho \left[\frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)\right] 4\pi r^2\,dr$$

we need only consider $r < R$
because $\rho = 0$ for $r > R$

$$= \frac{1}{2}\left(\frac{3q}{4\pi R^3}\right)\left(\frac{q}{2\epsilon_0 R}\right)\left[R^3 - \frac{R^3}{5}\right]$$

$$\boxed{W = \frac{3q^2}{20\pi\epsilon_0 R}}$$

(b) $W = \frac{\epsilon_0}{2}\int E^2 d\tau$

$$= \frac{\epsilon_0}{2}\int_0^R \left(\frac{qr}{4\pi\epsilon_0 R^3}\right)^2 4\pi r^2 dr + \frac{\epsilon_0}{2}\int_R^\infty \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$= \frac{q^2}{4\pi\epsilon_0 \times 2}\int_0^R \frac{r^4 dr}{R^6} + \frac{q^2}{4\pi\epsilon_0 \times 2}\int_R^\infty \frac{dr}{r^2}$$

$$= \frac{q^2}{8\pi\epsilon_0}\left[\frac{R^5}{5R^6} - \left(\cancel{\frac{1}{\infty}}^{0} - \frac{1}{R}\right)\right]$$

$$\boxed{W = \frac{3q^2}{20\pi\epsilon_0 R}}$$

(c) $W = \frac{\epsilon_0}{2}\left[\int_0^a E^2 d\tau + \oint V\vec{E}\cdot d\vec{S}\right]$

$$= \frac{q^2}{8\pi\epsilon_0}\left[\int_0^R \frac{r^4 dr}{R^6} + \int_R^a \frac{dr}{r^2}\right] + \frac{\epsilon_0}{2}\oint \frac{q}{4\pi\epsilon_0 a}\frac{qa^2 d\Omega}{4\pi\epsilon_0 a}$$

$$= \frac{q^2}{8\pi\epsilon_0}\left[\frac{1}{R} - \left(\frac{1}{a} - \frac{1}{R}\right)\right] + \frac{q^2}{8\pi\epsilon_0 a}$$

$$\boxed{W = \frac{3q^2}{20\pi\epsilon_0 R}, \text{ independent of } a}$$