(19) Find W for system with Yukawa Potential

$$V = k\frac{e^{-\alpha r}}{r}$$

Let's find W by building the charge configuration one shell at a time (See problem 2.33 in text).

From previous week's assignment,

$$\rho(r) = 4\pi\epsilon_0 k\,\delta^3(\vec{r}) - k\epsilon_0\frac{e^{-\alpha r}}{r}\alpha^2$$

we dropped $e^{-\alpha r}$ from $\uparrow$ because
$\delta(\vec{r}) = 0$ except at $\vec{r}=0$
$\Rightarrow e^{-\alpha r} = 1$

(1) For clarity let's imagine building the configuration by:

(a) Bringing in the charged cloud onto the point charge, Q_p.

(b) Bringing in the charged cloud onto "itself", layer by layer.

(a) Potential of point charge is

$$\boxed{V_p(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}}$$

In the "cloud" a shell of charge w/ radius R and thickness dR is

$$dq_c = \underbrace{(4\pi R^2 dR)}_{\text{Volume element}} \rho_c(R)$$

This leads to an amount of Work needed to bring in the shell given by

$$dW_{p,c} = dq_c V_p(R)$$

Sum up (integrate these contributions) to find

$$W_{p,c} = \int_0^\infty dq_c V_p(R) \qquad K = \frac{Q_p}{4\pi\epsilon_0}$$

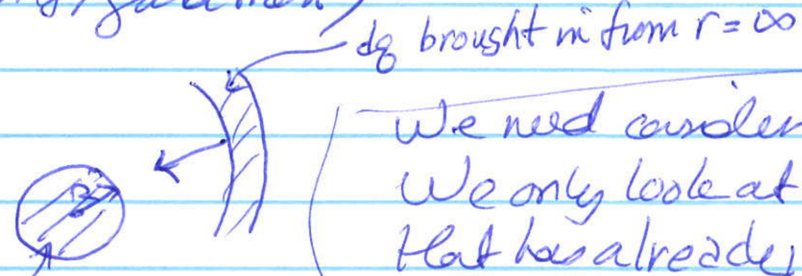
$$= \int_0^\infty \cancel{4\pi} R^2 dR \underbrace{\left[-\frac{Q_p}{\cancel{4\pi}}\alpha^2\frac{e^{-\alpha R}}{R}\right]}_{\rho_c(R)} \underbrace{\frac{Q_p}{4\pi\epsilon_0 R}}_{V_p(R)}$$

$$= -\frac{\alpha^2 Q_p^2}{4\pi\epsilon_0}\int_0^\infty e^{-\alpha R} dR$$

$$= \frac{\alpha^2 Q_p^2}{4\pi\epsilon_0}\left[\frac{e^{-\alpha R}}{-\alpha}\Big|_0^\infty\right]$$

$$\boxed{W_{p,c} = -\frac{\alpha Q_p^2}{4\pi\epsilon_0}}$$ ✓

(b) Potential of cloud (relevant part to build up configuration):



We need consider other charge. We only look at the charge that has already been brought in.

$$Q_c(R)=\int_0^R \rho(r)\,d^3x \implies \underbrace{V_c(R)=\frac{Q_c(R)}{4\pi\epsilon_0 R}}_{\text{"Relevant" potential}}$$

(i) find

$$Q_c(R)=\int_0^R \rho_c(r)\,d^3x$$

$$=\int_0^R \underbrace{\left[-\frac{Q_p\alpha^2}{\cancel{4\pi}}\frac{e^{-\alpha r}}{r}\right]}_{\rho_c(r)}\underbrace{\cancel{4\pi}r^2dr}_{d^3x}$$

$$=-\alpha^2 Q_p\int_0^R re^{-\alpha r}dr, \quad \text{do by parts}$$

$$=-\alpha^2 Q_p\left\{\frac{1}{\alpha^2}-\left(\frac{1}{\alpha^2}+\frac{R}{\alpha}\right)e^{-\alpha R}\right\}$$

and

$$\boxed{V_c(R)=-\frac{Q_p}{4\pi\epsilon_0 R}\left\{1-(1+\alpha R)e^{-\alpha R}\right\}}$$

$$W_{c,c} = \int_0^\infty dq_c(R)\, V_c(R)$$

$$= \int_0^\infty \left(-\frac{Q_p\alpha^2}{\cancel{4\pi}}\frac{e^{-\alpha R}}{R}\cancel{4\pi}R^2 dR\right)\left(-\frac{Q_p}{4\pi\epsilon_0 R}\left[1-(1+\alpha R)e^{-\alpha R}\right]\right)$$

$$= \frac{Q_p^2\alpha^2}{4\pi\epsilon_0}\int_0^\infty e^{-\alpha R}dR\left(1-e^{-\alpha R}-\alpha R e^{-\alpha R}\right)$$

$$= \frac{Q_p^2\alpha^2}{4\pi\epsilon_0}\left[-\frac{1}{\alpha}e^{-\alpha R}\Big|_0^\infty + \frac{e^{-2\alpha R}}{2\alpha}\Big|_0^\infty - \int_0^\infty \alpha R e^{-2\alpha R} dR\right]$$

$$= \frac{Q_p^2\alpha^2}{4\pi\epsilon_0}\left[\left(+\frac{1}{\alpha}-\frac{1}{2\alpha}\right)-\left\{\alpha\left(\frac{1}{4\alpha^2}\right)\right\}\right]$$

$$= \frac{Q_p^2\alpha^2}{4\pi\epsilon_0}\left[\frac{1}{2\alpha}-\frac{1}{4\alpha}\right]$$

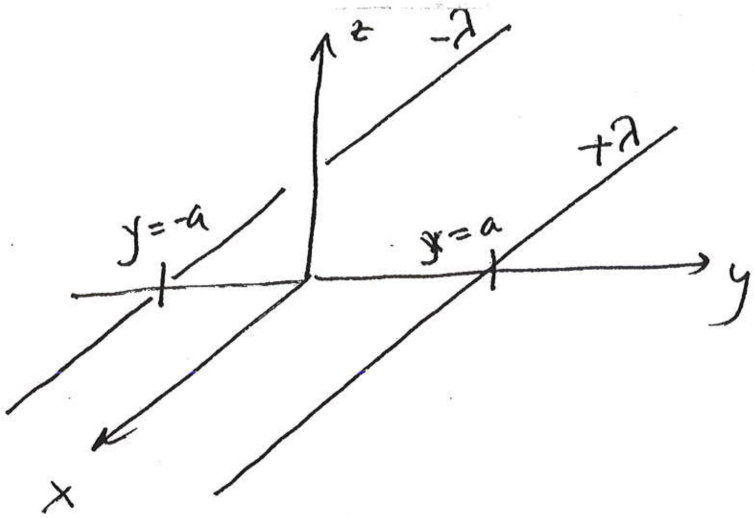
$$\boxed{W_{c,c} = \frac{Q_p^2\alpha}{16\pi\epsilon_0}}$$

Binding Energy is

$$W_{Total} = W_{p,c} + W_{c,c} = -\frac{Q_p^2\alpha}{4\pi\epsilon_0} + \frac{Q_p^2\alpha}{16\pi\epsilon_0}$$

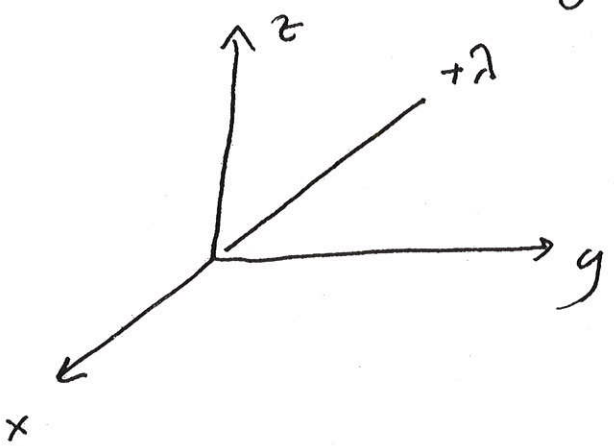
$$\boxed{W_{Total} = -\frac{3}{4}\frac{Q_p^2\alpha}{4\pi\epsilon_0}}$$

(20) Prob. 2.47



a) Find $V(x,y,z)$ using the origin as your reference

Consider 1 line charge along x-axis



By Gauss's law,

$$\vec{E}_s = \hat{s}\,\frac{\lambda}{2\pi\epsilon_0 s}$$

and $dV = -\vec{E}_s \cdot d\vec{s}$

$$V = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s}{s_0}\right)$$

s_0 ← reference pt.

Translate V to $y = \pm a$ (Use Cartesian coordinates)

(i) $V(x,y,z) = -\frac{\lambda}{2\pi\epsilon_0}\ln\left[\frac{\sqrt{y^2+z^2}}{s_0}\right]$ is "x-axis sol$^{\underline{n}}$"

Translate to $y = \pm a$ and sum for 2 line charges,

$$V(x,y,z) = -\frac{\lambda}{2\pi\epsilon_0}\left\{\ln\left[\frac{\sqrt{(y-a)^2+z^2}}{s_0}\right] - \ln\left[\frac{\sqrt{(y+a)^2+z^2}}{s_0}\right]\right.$$

2$^{\underline{nd}}$ line has $\lambda' = -\lambda$

$$\boxed{V(x,y,z) = -\frac{\lambda}{2\pi\epsilon_0}\ln\left[\frac{\sqrt{(y-a)^2+z^2}}{\sqrt{(y+a)^2+z^2}}\right]}$$

b) Show that the equipotentials are circular cylinders and locate the axis and radius of the cylinder corresponding to a given potential V_0.

$V(x,y,z) = V_0 \Rightarrow$ defines equipotential surfaces.

$$\Rightarrow -\frac{2\pi\epsilon_0 V_0}{\lambda} = \ln\left[\frac{\sqrt{(y-a)^2+z^2}}{\sqrt{(y+a)^2+z^2}}\right]$$

$$-\frac{4\pi\epsilon_0 V_0}{\lambda} = \ln\left[\frac{(y-a)^2+z^2}{(y+a)^2+z^2}\right] \leftarrow \text{took square root outside } \underline{\ln}.$$

$$\underset{\text{constant}}{C_0} = \exp\left(-\frac{4\pi\epsilon_0 V_0}{\lambda}\right) = \left(\frac{(y-a)^2+z^2}{(y+a)^2+z^2}\right)$$

$$\Rightarrow C_0\left[(y+a)^2+z^2\right] = (y-a)^2+z^2$$

$$z^2(C_0-1) + \left\{y^2(C_0-1) + y\left(2aC_0+2a\right) + a^2(C_0-1)\right\} = 0$$

$$z^2 + \underbrace{\left\{y^2 + y\,2a\left(\frac{C_0+1}{C_0-1}\right) + a^2\right\}}_{\text{complete the square}} = 0$$

$$y^2 + 2a\left(\frac{C_0+1}{C_0-1}\right)y + a^2 = 0$$

$$\left(y + a\left[\frac{C_0+1}{C_0-1}\right]\right)^2 + \left(a^2 - a^2\left[\frac{C_0+1}{C_0-1}\right]^2\right) = 0$$

$$\left(y - a\left[\frac{1+C_0}{1-C_0}\right]\right)^2 + a^2\left(\frac{C_0^2-2C_0+1-C_0^2-2C_0-1}{(C_0-1)^2}\right) = 0$$

$$\left(y - a\left[\frac{1+C_0}{1-C_0}\right]\right)^2 + a^2\left(\frac{-4C_0}{(C_0-1)^2}\right) = 0$$

So, we have

$$\boxed{\left(y - a\left[\frac{1+C_0}{1-C_0}\right]\right)^2 + z^2 = \frac{4a^2C_0}{(C_0-1)^2}}$$

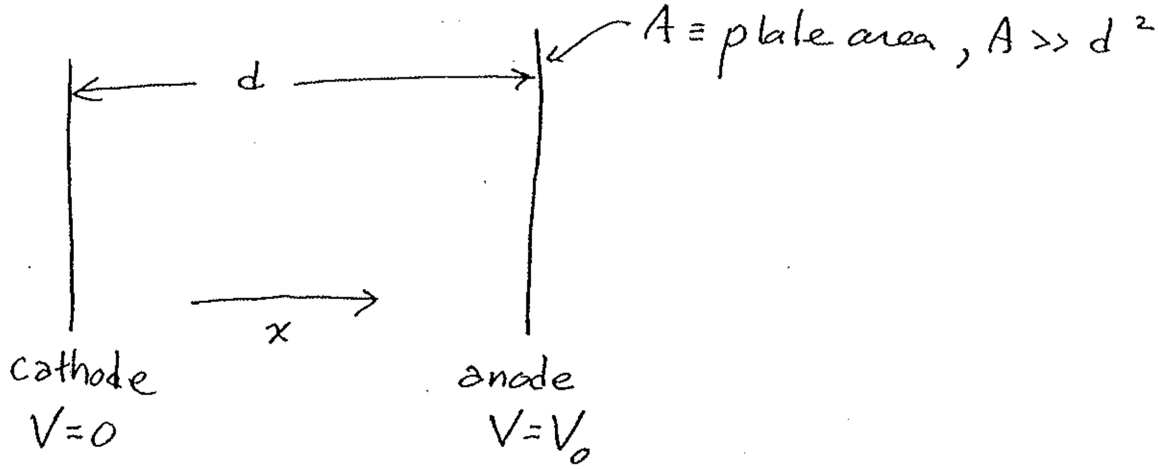
the above is an equation for a circle centered at

$$y = a\left(\frac{1+C_0}{1-C_0}\right) \quad \text{and} \quad z = 0$$

with radius

$$R = \frac{2a}{|C_0-1|}\sqrt{C_0}$$

(21) Prob. 2.48



a) Poisson's Equation:

$$\boxed{\frac{\partial^2}{\partial x^2} V = -\rho(x)/\epsilon_0}$$

b) $E = \frac{1}{2} m v^2 + q V(x)$: Energy equation for e^-'s

at $x=0$, $V=0$, $v^2=0$ $\rightarrow$ $E=0$

$$\Longrightarrow \frac{1}{2} m v^2 = -q V(x) \rightarrow v^2(x) = -\frac{2q}{m} V(x) \quad q=-e$$

$$\boxed{v^2(x) = \frac{2e}{m} V(x)}$$

c) In steady state, I_0 is independent of x. What is the relation between ρ and $v(x)$?

$$J_0 = \frac{I_0}{A} = \rho v \Rightarrow \boxed{I_0 = \rho v A = \text{constant} < 0}$$

e^-'s move to right

d) Using a, b, c find a differential equation for V.

$$\frac{\partial^2}{\partial x^2} V(x) = -\frac{1}{\epsilon_0}\left[\frac{I_0}{vA}\right] \text{, substituting for } \rho$$

$$= -\frac{I_0}{\epsilon_0 A}\left[\frac{m}{2eV(x)}\right]^{1/2} \text{, substituting for } v(x) \text{ velocity}$$

$$\boxed{\frac{\partial^2}{\partial x^2} V(x) + \sqrt{\frac{m I_0^2}{\epsilon_0^2 A^2 2e}}\, V^{-1/2}(x) = 0}$$

e) find $V(x)$ as a function of x, V_0, and d

Guess that

$$V(x) = a x^b$$

is a solution to the differential equation

$$a b(b-1) x^{b-2} + \sqrt{\frac{m I_0^2}{\epsilon_0^2 A^2 2e}} \frac{1}{a^{1/2}} x^{-b/2} = 0$$

$$a b(b-1) x^{\frac{3b}{2}-2} + \sqrt{\frac{m I_0^2}{2\epsilon_0^2 A^2 e}} \frac{1}{\sqrt{a}} = 0$$

$$\Longrightarrow b = 4/3$$

and so,

$$\frac{4}{3} a\left(\frac{1}{3}\right) + \frac{1}{\sqrt{a}} \sqrt{\frac{m I_0^2}{2\epsilon_0^2 A^2 e}} = 0$$

$$\Longrightarrow a^{3/2} = -\frac{9}{4} \frac{I_0}{\epsilon_0 A} \sqrt{\frac{m}{2e}}$$

the potential is

$$V(x) = \left[\frac{9}{4} \frac{I_0}{\epsilon_0 A} \sqrt{\frac{m}{2e}}\right]^{\frac{2}{3}} x^{4/3}$$

Hmm, we want $V(x)$ in terms of x, V_0, and d. Let

$$V(x) = +K x^{4/3}, \text{ where } K = \left[\frac{9 I_0}{4\epsilon_0 A} \sqrt{\frac{m}{2e}}\right]^{2/3}$$

at $x = d$, $V = V_0$

$$V_0 = +Kd^{4/3} \longrightarrow +K = V_0/d^{4/3}$$

and

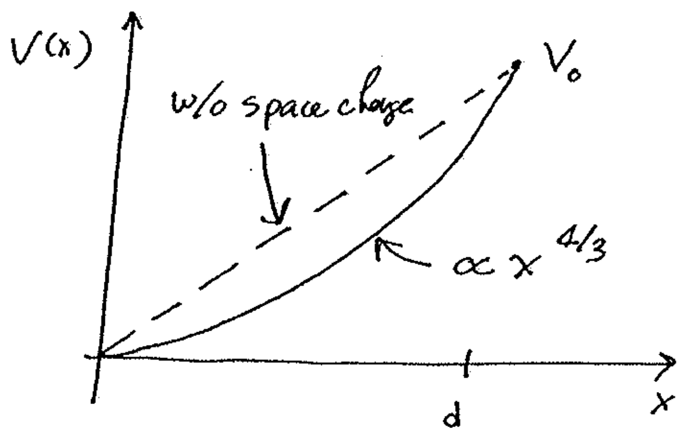
$$\boxed{V(x) = \left(\frac{V_0}{d^{4/3}}\right) x^{4/3}}$$

find $\rho(x)$ and $v(x)$

$$v^2(x) = \frac{2e}{m}\left(\frac{V_0}{d^{4/3}}\right) x^{4/3}$$

$$\rho(x) = \left(\frac{I_0}{A}\right)\left[\frac{2e}{m}\frac{V_0}{d^{4/3}}\right]^{-\frac{1}{2}} x^{-2/3}$$

Plot $v(x)$



f) Show that $I_0 = K V_0^{3/2}$ and find K.

$$V(x) = \left[\frac{9 I_0}{4\epsilon_0 A}\sqrt{\frac{m}{2e}}\right]^{2/3} x^{4/3}$$

at $x = d$, $V = V_0$

$$V_0 = \left[\frac{9 I_0}{4\epsilon_0 A}\sqrt{\frac{m}{2e}}\right]^{2/3} d^{4/3}$$

$$\Rightarrow \underbrace{I_0}_{>0} = \underbrace{d^{-2}\left(\frac{4\epsilon_0 A}{9}\sqrt{\frac{2e}{m}}\right)}_{K} V_0^{3/2}$$

(we defined I_0 in [illegible])

2.49

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\hat{r}$$

(a) What is the electric field of a charge distribution ρ?

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, d^3x}{\mathscr{r}^2}\left(1+\frac{\mathscr{r}}{\lambda}\right)e^{-\frac{\mathscr{r}}{\lambda}}\hat{\mathscr{r}}}$$

(b) Does this electric field admit a scalar potential?

$\boxed{\text{Yes,}}$ because

$$\vec{\nabla}\times\vec{E} = \frac{1}{4\pi\epsilon_0}\vec{\nabla}\times\underbrace{\int \frac{\rho\, d^3x}{\mathscr{r}^2}\left(1+\frac{\mathscr{r}}{\lambda}\right)e^{-\mathscr{r}/\lambda}\hat{\mathscr{r}}}_{\text{radial function because}}$$

radial function because

$$\hat{\mathscr{r}} = \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\ ;\ \vec{r}\,' \text{ integration variable}$$

and so, $\vec{\nabla}\times f(\mathscr{r})\hat{\mathscr{r}} = 0$

(c) $$-dV = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\hat{r}\cdot d\vec{r}$$

$$-V(r)+V_\infty = \frac{q}{4\pi\epsilon_0}\int_\infty^r \frac{(1+r/\lambda)}{r^2}e^{-r/\lambda}dr$$

$$\cancel{= \frac{q}{4\pi\epsilon_0}\left[-\frac{q}{r}e^{-r/\lambda} + \frac{1}{\lambda}\left(\log r + \frac{r}{\lambda} + \frac{r^2}{4\lambda^2} + \frac{r^3}{18\lambda^3} + \dots\right)\right]}$$

$$= \frac{q}{4\pi\epsilon_0}\left[-\frac{e^{-r/\lambda}}{r} - \frac{1}{\lambda}\int_\infty^r \frac{e^{-r/\lambda}}{r}dr + \int_\infty^r \frac{e^{-r/\lambda}}{\lambda r}dr\right]$$

$$\boxed{V(r) = \underbrace{V_\infty}_{\to 0} + \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r} = \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r}}$$