$$E_r(r=R) = \frac{-q}{4\pi\epsilon_0 R}\left[\frac{a^2-R^2}{(a^2+R^2-2aR\cos\theta)^{3/2}}\right] = \frac{\sigma(\theta)}{\epsilon_0}$$

$$\Rightarrow \boxed{\sigma(\theta) = \frac{-q}{4\pi R}\left[\frac{a^2-R^2}{(a^2+R^2-2aR\cos\theta)^{3/2}}\right]}$$

c) Integrate over the spherical surface

$$Q = \int \sigma R^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{-q}{\underset{2}{\cancel{4\pi}} R}(a^2-R^2)\cancel{2\pi}\int R^2 \sin\theta \, d\theta \frac{1}{(a^2+R^2-2aR\cos\theta)^{3/2}}$$

let: $u = a^2+R^2-2aR\cos\theta \Rightarrow du = 2aR\sin\theta \, d\theta$

$$= \frac{-q(a^2-R^2)\cancel{R}}{4a\cancel{R}}\int_{(a-R)^2}^{(a+R)^2} \frac{du}{u^{3/2}}$$

$$= \frac{-(a^2-R^2)}{4a} q \left[-2\left\{\frac{1}{a+R} - \frac{1}{a-R}\right\}\right]$$

$$= \frac{\cancel{(a^2-R^2)}\,q}{-\cancel{2}a}\left[\frac{+\cancel{2}R}{\cancel{a^2-R^2}}\right]$$

$$\boxed{Q = -\left(\frac{R}{a}\right)q}$$

(d) Calculate energy of the configuration

recall $\left\{ \begin{array}{l} q' = -\left(\frac{R}{a}\right) q \\ b = \left(\frac{R}{a}\right) R, \text{ where } a \text{ is the distance to the charge } q \end{array} \right.$

$$\Rightarrow \vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q\left(-\frac{Rq}{r}\right)\hat{r}}{\left[r - \frac{R^2}{r}\right]^2}$$

$\leftarrow r\hat{r}\cdot d\vec{r} = d\left(\frac{1}{2}r^2\right)$

$$W = -\frac{q^2 R}{4\pi\epsilon_0}\int_{\infty}^{a} \frac{r\hat{r}}{(r^2 - R^2)^2}\cdot d\vec{r}$$

$$= -\frac{q^2 R}{4\pi\epsilon_0}\int_{\infty}^{a} \frac{\frac{1}{2}dr^2}{(r^2 - R^2)^2}$$

$$= -\frac{q^2 R}{8\pi\epsilon_0}\left[\frac{-1}{(r^2 - R^2)}\right]_{\infty}^{a}$$

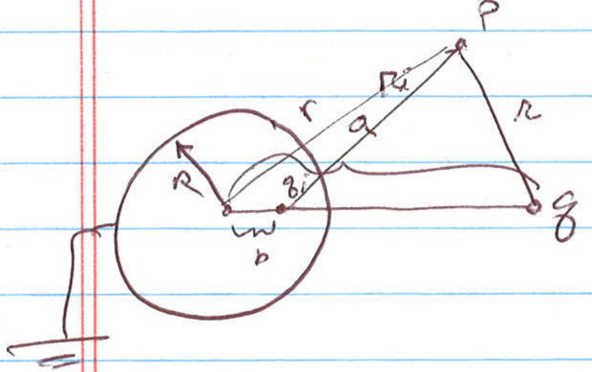
$$\boxed{W = \frac{q^2}{8\pi\epsilon_0}\left[\frac{R}{(a^2 - R^2)}\right]}$$

re-express this using q' & b (for amusement)

$$W = \frac{q}{8\pi\epsilon_0}\left[-\frac{a}{\cancel{R}}q'\right]\frac{\cancel{R}}{(a^{\cancel{2}} - \cancel{a}b)}$$

$$= -\frac{qq'}{8\pi\epsilon_0(a-b)}$$

Problem 3.8



a) $\frac{q_i}{4\pi\epsilon_0 r_i}$ & $\frac{q}{4\pi\epsilon_0 r}$ are chosen to make $V_{total}=0$ when $r=R$

b) if we add q_i' to the center of the sphere, $V=V_0$ at $r=R$ and we do not disturb the solution

c) we want $V_0 = \frac{1}{4\pi\epsilon_0}\frac{q_i'}{R}$

$$\Rightarrow \boxed{q_i' = 4\pi\epsilon_0 R V_0 \text{ at } r=0}$$

(d) Find the force of attraction between a point charge q and a neutral conducting sphere

neutral sphere

if we place an image charge at $b = R\left(\frac{R}{a}\right)$ with charge $q' = -\left(\frac{R}{a}\right)q$

- then $\vec{E}=0$ in the conductor and $V=0$ at surface of conductor.
- We can "neutralize" the sphere by adding an

image charge at the origin of size

$$\boxed{q'' = -q' = \left(\frac{R}{a}\right)q}$$

The force on q is then

image charge 1

image charge at origin

$$\vec{F} = \frac{1}{4\pi\epsilon_0} q \left[\frac{-\left(\frac{R}{a}\right)q}{\left[a - \frac{R^2}{a}\right]^2} + \frac{\left(\frac{R}{a}\right)q}{a^2} \right] \hat{z}$$

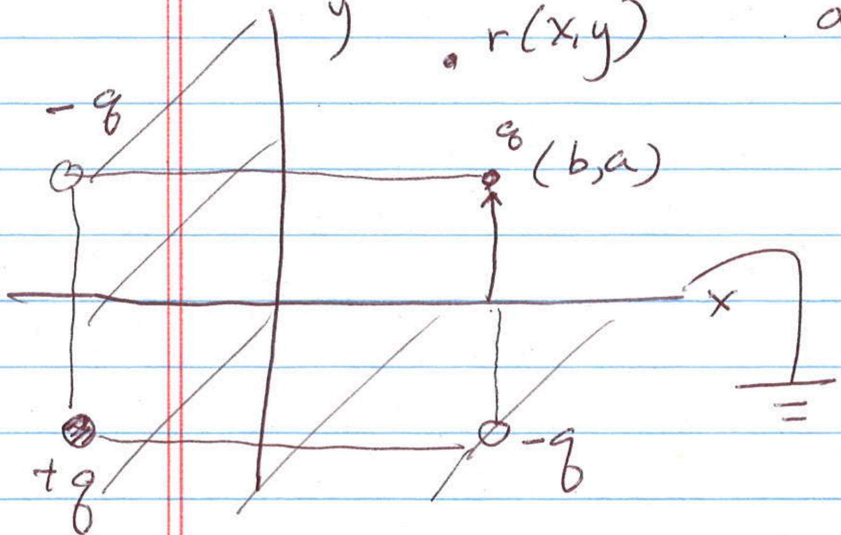
The force of attraction is smaller than on the grounded conducting sphere.

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \left[\frac{R}{a} q^2\right] \times \left[\frac{1}{a^2} - \frac{a^2}{(a^2-R^2)^2}\right] \hat{z}$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{Rq^2}{a}\right] \cdot \left[\frac{\cancel{a^4} - 2a^2R^2 + R^4 - \cancel{a^4}}{a^2(a^2-R^2)^2}\right] \hat{z}$$

$$\boxed{\vec{F} = \frac{q^2}{4\pi\epsilon_0} \left[\frac{R^3}{a^3}\right] \left[\frac{R^2 - 2a^2}{(a^2-R^2)^2}\right] \hat{z}}$$

Problem 3.10



a)

Charge	(x, y)
$-q$	$(b, -a)$
$-q$	$(-b, a)$
$+q$	$(-b -a)$

b) find force on q. Use the potential at $r(x,y)$.

$$V(x,y) = \underbrace{\frac{q}{4\pi\epsilon_0}\left[\frac{-1}{\sqrt{(x-b)^2+(y+a)^2}} + \frac{-1}{\sqrt{(x+b)^2+(y-a)^2}} + \frac{1}{\sqrt{(x+b)^2+(y+a)^2}}\right]}_{\text{image charges}}$$

$$+ \underbrace{\frac{q}{4\pi\epsilon_0\sqrt{(x-b)^2+(y-a)^2}}}_{\text{point charge}}$$

c) field due to the image charges is

$$\vec{E}_{image} = \frac{q}{4\pi\epsilon_0}\left\{\frac{-(x-b)\hat{x} - (y+a)\hat{y}}{\left[(x-b)^2+(y+a)^2\right]^{3/2}} + \frac{-(x+b)\hat{x} - (y-a)\hat{y}}{\left[(x+b)^2+(y-a)^2\right]^{3/2}} + \frac{+(x+b)\hat{x} + (y+a)\hat{y}}{\left[(x+b)^2+(y+a)^2\right]^{3/2}}\right\}$$

and so the force is $\vec{F} = q\vec{E}_{image}$

charge q is at $(x, y) = (b, a)$

$$\boxed{\vec{F}_{\text{on } q} = \frac{q^2}{4\pi\epsilon_0}\left\{\frac{-2a\hat{y}}{(4a^2)^{3/2}} - \frac{2b\hat{x}}{(4b^2)^{3/2}} + \frac{2b\hat{x} + 2a\hat{y}}{(4b^2+4a^2)^{3/2}}\right\}}$$

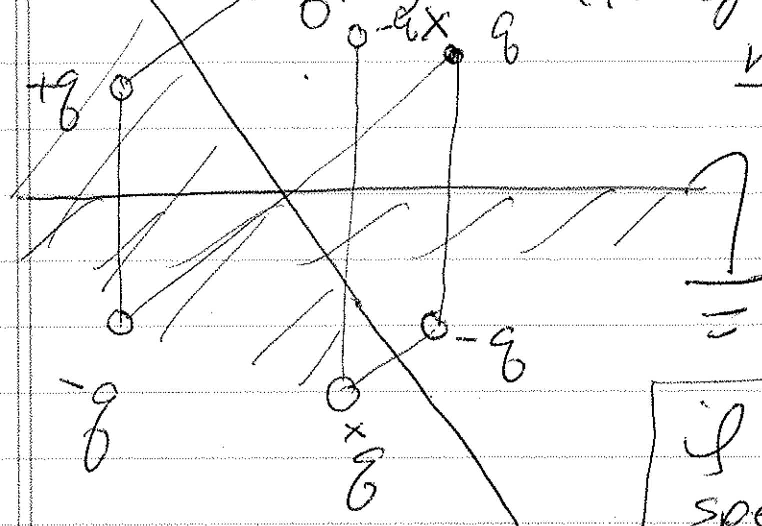
(d) Suppose the plates met at some $\measuredangle$ other than $90°$, would you still be able to solve the problem by the method of images?

not in general

(e) If not, for what particular $\measuredangle$s does the method work?

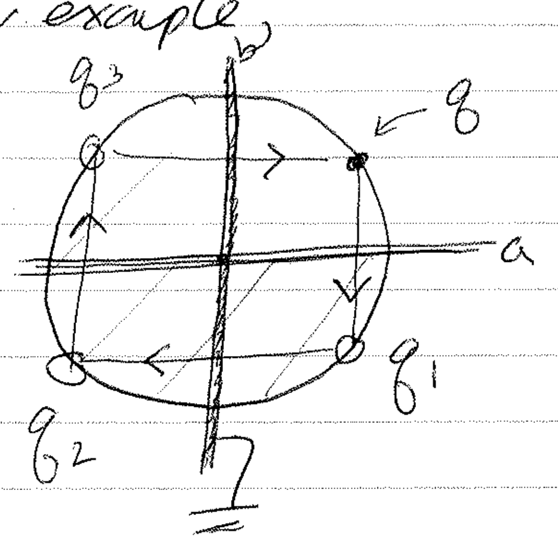
(a) First off, if the ∡ of intersection is $> 90^\circ$

$-q$ X, need an image charge in vacuum region, not allowed



if ∡ $> 90^\circ$ (except for special case of 180°) can't use image charges

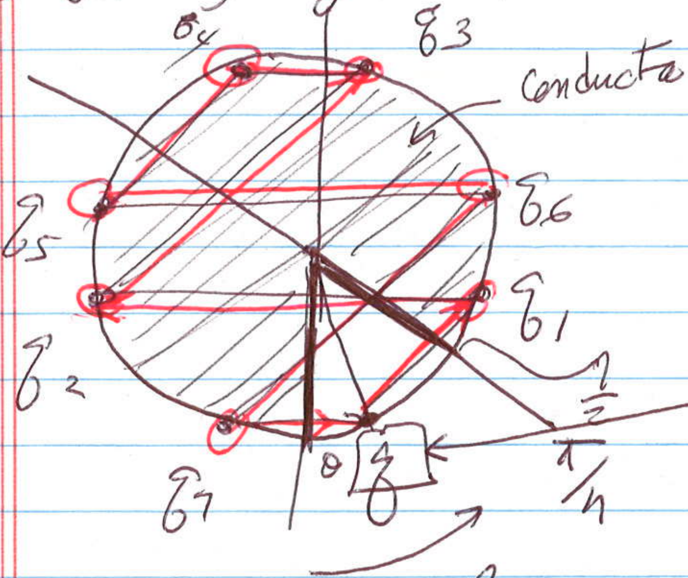
(b) For ∡ 90° show that if π/n is the opening ∡, then we can find an exact # of image charges.
For example,



(i) 2 plates intersect at an angle of 90° $(\pi/2)$
(ii) q_1 makes $V=0$ on plane a
(iii) q_2 cancels q_1 on plane b
(iv) q_3 cancels q_2 on plane a
(v) in a sense, q_3 cancels q_3 on plane b

closes

(c) for any integer n, we can have



angle increases counter-clockwise

if n is an integer, this construction closes on the charge q

charge locations are

	q	θ
	q_1	$2\pi/n - \theta$
$n=2$	q_2	$-q_1$
	q_3	$4\pi/n - \theta$
$n=3$	q_4	$-q_3$
$n=4$	q_5	$6\pi/n - \theta$
	q_6	$-q_5$
$n=5$	q_7	$8\pi/n - \theta$
	q_8	$-q_7$
	q_9	$10\pi/n - \theta$
	q_{10}	$-q_9$
	⋮	⋮

we need to keep $(2n-1)$ image charges.

If n is <u>not</u> an integer, doesn't close. There is a solution but is <u>not</u> conveniently found using image charges