Physics 412: Introduction to Electrodynamics
Homework ~~5~~ 6
Due: Friday, 2012 November 9

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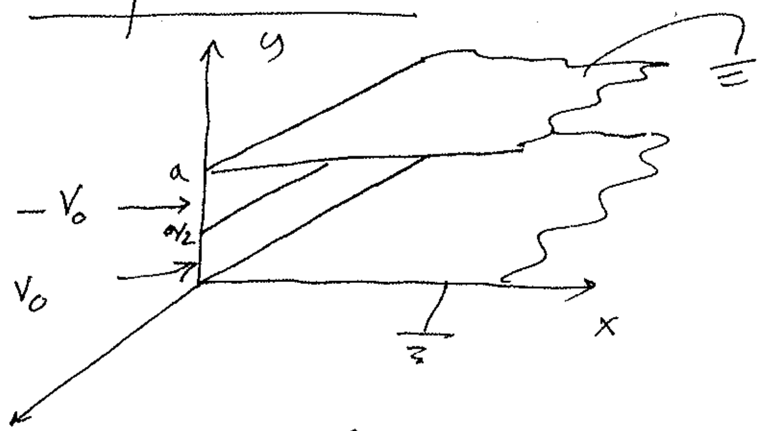
Prob 3.12

Find V in the infinite slot if the boundary at $x=0$ consists of 2 strips: one from $y=0$ to $y=a/2$, is held at constant potential V_0, and the other, from $y=a/2$ to a, is at potential $-V_0$



Use sol^n from ex 3.3 (up to imposition of $x=0$ BC)

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a}y\right) \qquad (3.30)$$

at $x=0$

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}y\right) = V_0(x=0,y)$$

multiply by $\sin\left(\frac{n'\pi}{a}y\right)dy$ and integrate over $y=[0,a]$

$$\int_0^a \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right)dy = \int_0^{\frac{a}{2}} V_0 \sin\left(\frac{n'\pi}{a}y\right)dy - \int_{\frac{a}{2}}^{a} V_0 \sin\left(\frac{n'\pi}{a}y\right)dy$$

$$\frac{a}{2}C_{n'} = V_0\frac{a}{n'\pi}\left[-\cos\left(\frac{n'\pi}{a}y\right)\Big|_0^{a/2} + \cos\left(\frac{n'\pi}{a}y\right)\Big|_{a/2}^{a}\right]$$

$$= \frac{aV_0}{n'\pi}\left[\left(-\cos\left(\frac{n'\pi}{2}\right)+1\right) + \left(\cos(n'\pi) - \cos\left(\frac{n'\pi}{2}\right)\right)\right]$$

$$= \frac{aV_0}{n'\pi}\left[1 + \cos(n'\pi) - 2\cos\left(\frac{n'\pi}{2}\right)\right]$$

$$\frac{a}{2} C_{n'} = \frac{aV_0}{n'\pi}\left[\left(\cos(n'\pi)+1\right) - 2\cos\left(\frac{n'\pi}{2}\right)\right]$$

$$= \frac{aV_0}{n'\pi}\begin{cases} 0 & n' = \text{odd}\,;\ 4, 8, 12, 16, \cdots \\ 4 & n' = 2, 6, 10, \cdots \\ & \quad = (4n+2),\ n = 0, 1, 2, 3, \cdots \end{cases}$$

$$\Longrightarrow C_{n'} = \frac{8V_0}{n'\pi}, \quad n' = (4n+2) \text{ where } n = 0, 1, 2, 3, \ldots$$

or

$$C_n = \frac{8V_0}{(4n+2)\pi}, \quad n = 0, 1, 2, \cdots$$

$$\boxed{V(x,y) = \sum_{n=0}^{\infty} \frac{8V_0}{(4n+2)\pi} e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a}y\right)}$$

Prob 3.13

For the infinite slot in Ex 3.3, determine $\sigma(y)$ at $x=0$ assuming it has a constant potential V_0

Use the "summed" sol^{n} for Ex 3.3

$$V(x,y) = \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right)$$

to find $\vec{E}(x=0,y)$ and $\therefore$ $\sigma(y) = \epsilon_0 E_x$

$$E_x = -\frac{\partial V(x,y)}{\partial x} = \frac{-2V_0}{\pi}\left[\frac{1}{1+\left\{\frac{\sin(\pi y/a)}{\sinh(\frac{\pi x}{a})}\right\}^2}\right] \times \left[\frac{-\sin(\frac{\pi y}{a})}{\sinh^2(\frac{\pi x}{a})} \frac{\pi}{a} \cosh \frac{\pi x}{a}\right]$$

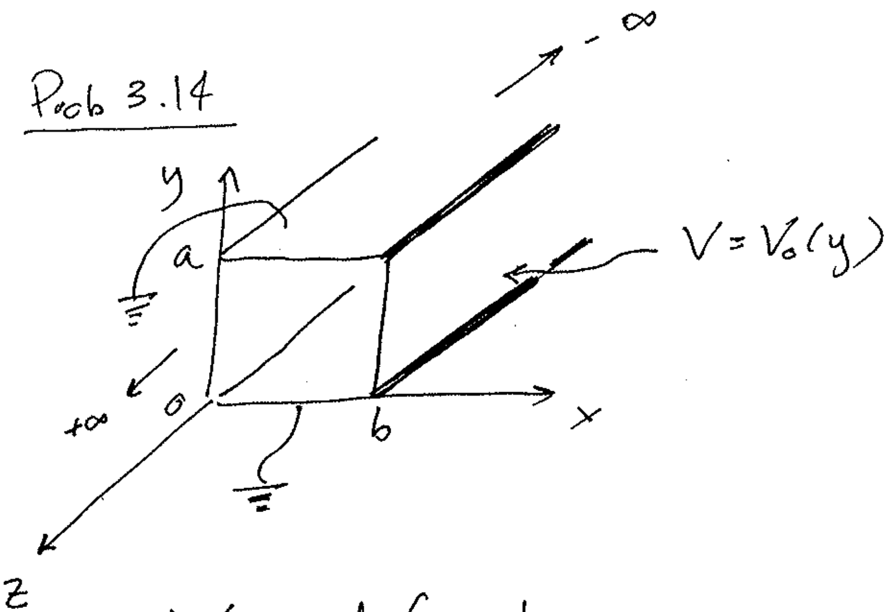
at $x=0$

$$E_x = \frac{2V_0}{a} \sin\left(\frac{\pi y}{a}\right) \frac{1}{1+\sin^2(\frac{\pi y}{a})}$$

and

$$\boxed{\sigma(y) = \frac{2V_0\epsilon_0}{a}\left[\frac{\sin(\frac{\pi y}{a})}{1+\sin^2(\frac{\pi y}{a})}\right]}$$

Prob 3.14



B.C.'s

① $V = V_0(y)$ at $x = b$

② $V = 0$ at $y = 0, a$
$x = 0$

a) General formula

"Basic" Solution is

$$V(x,y,z) = \left(Ce^{+\gamma_{\alpha\beta}x} + De^{-\gamma_{\alpha\beta}x}\right)\left(A\cos(\alpha y) + B\sin(\alpha y)\right) \times \left(E\cos(\beta z) + F\sin(\beta z)\right)$$

Now consider BC's and symmetries

(i) Because the pipe is infinite in $z \Rightarrow \frac{\partial V}{\partial z} = 0 \Rightarrow \beta = 0, E = 1$

$$V(x,y) = \left(A\cos(\alpha y) + B\sin(\alpha y)\right)\left(Ce^{\gamma_\alpha x} + De^{-\gamma_\alpha x}\right)$$

note: $\beta = 0 \rightarrow \gamma_\alpha = \alpha$

(ii) at $x = 0$, $V = 0 \Rightarrow C = -D$ and let $C = 1$

$$V(x,y) = \left(A\cos\alpha y + B\sin\alpha y\right)\sinh(\alpha x)$$

(iii) at $y = 0$, $V = 0 \Rightarrow A = 0$

at $y = a$, $V = 0 \Rightarrow \alpha = \frac{n\pi}{a}$

$$V(x,y) = A\sin\left(\frac{n\pi}{a}y\right)\sinh\left(\frac{n\pi}{a}x\right)$$

(iv) at $x=b$, $V=V_0(y)$

$$V_0(y) = A\sin\left(\frac{n\pi}{a}y\right)\sinh\left(\frac{n\pi b}{a}\right)$$

multiply by $\sin\left(\frac{n'\pi}{a}y\right)dy$ and integrate over $y=[0,a]$

$$\int_0^a V_0(y)\sin\left(\frac{n'\pi}{a}y\right)dy = \int_0^a \sum_{n=0}^{\infty} C_n\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right)dy\,\sinh\left(\frac{n\pi}{a}b\right)$$

$$= \frac{a}{2}C_{n'}\sinh\left(\frac{n'\pi}{a}b\right)$$

$$\Rightarrow C_{n'} = \frac{2}{a\sinh\left(\frac{n'\pi}{a}b\right)}\int_0^a V_0(y)\sin\left(\frac{n'\pi}{a}y\right)dy$$

and $$\boxed{V(x,y) = \sum_{n=0}^{\infty} C_n\sin\left(\frac{n\pi}{a}y\right)\sinh\left(\frac{n\pi}{a}x\right)}$$

b) <u>let $V_0(y) = V_0 \equiv$ constant</u>

$$\frac{a}{2}C_{n'}\sinh\left(\frac{n'\pi b}{a}\right) = V_0\int_0^a \sin\left(\frac{n'\pi}{a}y\right)dy$$

$$= -V_0\left[\cos(n'\pi) - 1\right]\frac{a}{n'\pi}$$

$$= -V_0\left[\begin{matrix}-2 & n' \text{ odd}\\ 0 & n' \text{ even}\end{matrix}\right]\frac{a}{n'\pi}$$

$$\Rightarrow C_{n'} = \begin{cases}\dfrac{4V_0}{n'\pi\sinh\left(\frac{n'\pi}{a}b\right)}, & n' \text{ odd}\\ 0, & n' \text{ even}\end{cases}$$

$$\boxed{V(x,y) = \sum_{n \text{ odd}}^{\infty} C_n\sin\left(\frac{n\pi}{a}y\right)\sinh\left(\frac{n\pi}{a}x\right)}$$

Prob 3.16

Derive $P_3(\mu)$ from the Rodrigues formula & check that $P_3(\mu)$ satisfies the angular portion of the Laplace equation. Verify that $P_3(\mu)$ and $P_1(\mu)$ are orthogonal.

a) $$P_\ell(\mu) = \frac{1}{2^\ell \ell!}\left(\frac{d}{d\mu}\right)^\ell (\mu^2-1)^\ell$$

$$P_3(\mu) = \frac{1}{2^3 3!}\frac{d^3}{d\mu^3}(\mu^2-1)^3$$

$$= \frac{1}{8\times 6}\frac{d^2}{d\mu^2}\left(3\left[\mu^2-1\right]^2 2\mu\right)$$

$$= \frac{3}{48}\frac{d}{d\mu}\left[2(\mu^2-1)^2 + 4\mu(\mu^2-1)2\mu\right]$$

$$= \frac{1}{8}\left[2(\mu^2-1)2\mu + 8\mu(\mu^2-1) + 4\mu^2(2\mu)\right]$$

$$= \left[\frac{1}{2}(\mu^2-1)\mu + (\mu^2-1)\mu + (\mu^3)\right]$$

$$= \left[\frac{5}{2}\mu^3 - \frac{3}{2}\mu\right]$$

$$\boxed{P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu)}$$

b) $$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin\theta\,\Theta = 0 \qquad (3.60)$$

$$\frac{d}{d\theta}\left[\left(\sin\theta\frac{d}{d\theta}\right)\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right)\right] + 12\sin\theta\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) = 0$$

$$\frac{d}{d\theta}\left(\sin\theta\left\{-\frac{15}{2}\cos^2\theta\sin\theta + \frac{3}{2}\sin\theta\right\}\right) + 30\sin\theta\cos^3\theta - 18\sin\theta\cos\theta = 0$$

$$\frac{d}{d\theta}\left(-\frac{15}{2}\cos^2\theta\sin^2\theta + \frac{3}{2}\sin^2\theta\right) + 30\sin\theta\cos^3\theta - 18\sin\theta\cos\theta = 0$$

$$+15\cos\theta\sin^3\theta - 15\cos^3\theta\sin\theta + 3\sin\theta\cos\theta + 30\sin\theta\cos^3\theta - 18\sin\theta\cos\theta = 0$$

$$\sin\theta\cos\theta\left(15\sin^2\theta - \quad \cos^2\theta + 3 + 30\cos^2\theta - 18\right) = 0$$

$$\sin\theta\cos\theta\left[\underbrace{15 - 15\cos^2\theta - 15\cos^2\theta + 3 + 30\cos^2\theta - 18}\right] = 0$$

$\longrightarrow 0$, as required

c) $\int_{-1}^{1} P_1(\mu) P_3(\mu)\, d\mu$

$= \int_{-1}^{1} \mu \frac{1}{2}(5\mu^3 - 3\mu)\, d\mu$

$= \frac{1}{2}\int_{-1}^{1}(5\mu^4 - 3\mu^2)\, d\mu$

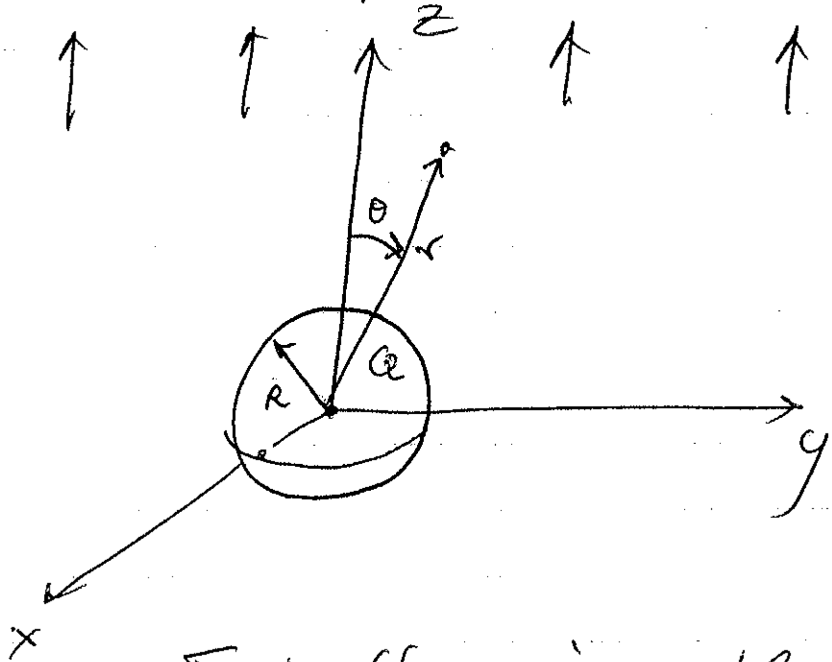
$= \frac{1}{2}\left(\frac{5}{5}\mu^5 - \frac{3}{3}\mu^3\right)\Big|_{-1}^{1}$

$= \frac{1}{2}\left(1 - (-1) - 1 + (-1)\right)$

$= 0$

Prob 3.20

Find Φ outside a charged metal sphere (charge Q, Radius R) placed in an otherwise uniform Electric field $\vec{E}_0$.



$\vec{E}_z = E_0 \hat{z}$

$\Rightarrow \Phi_\infty = -E_0 z + \Phi_0$

First off, consider the grounded metal sphere case $\Rightarrow \Phi = \Phi_R$ at $r = R$.

$$\Phi = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\mu)$$

at $r = \infty$

$$\Phi = -E_0 z + \Phi_0 = -E_0 r \mu + \Phi_0$$

$$\Rightarrow A_l = 0 \quad \text{if } l > 1$$

and so, $\Phi = \left(A_0 - E_0 r \overset{P_1(\mu)}{\mu} \right) + \sum_l \frac{B_l}{r^{l+1}} P_l(\mu)$

at $r = R$, $\Phi = \Phi_R$

$$\Rightarrow \quad \Phi = (A_0 - E_0 R \mu) + \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\mu) = \Phi_R$$

so (i) $B_\ell = 0$, $\ell > 1$

(ii) $A_0 + \frac{B_0}{R} = \Phi_R \Rightarrow A_0 = \Phi_R - B_0/R$

(iii) $-E_0 R + B_1/R^2 = 0 \Rightarrow B_1 = E_0 R^3$

and

$$\Phi = \left[\Phi_R - \frac{B_0}{R} - E_0 r \mu\right] + \frac{B_0}{r} + \frac{E_0 R^3}{r^2}\mu$$

Q: what is B_0?

A: Use charge on sphere Q to set B_0

at $r = R$, $\frac{\sigma}{\epsilon_0} = E_n$

$$\frac{\sigma}{\epsilon_0} = -\left[-E_0\mu - \frac{B_0}{R^2} - 2\frac{E_0 R^3}{R^3}\mu\right] = 3E_0\mu + \frac{B_0}{R^2}$$

$$\sigma = \epsilon_0\left[3E_0\mu + B_0/R^2\right]$$

integrates to 0

$$Q = \oint \sigma dS = 2\pi R^2 \epsilon_0 \int_1^{-1} \left(\cancel{3E_0\mu} + \frac{B_0}{R^2}\right) d(-\mu)$$

$$= 4\pi\epsilon_0 B_0 \Rightarrow B_0 = (Q/4\pi\epsilon_0)$$

So, the solution becomes

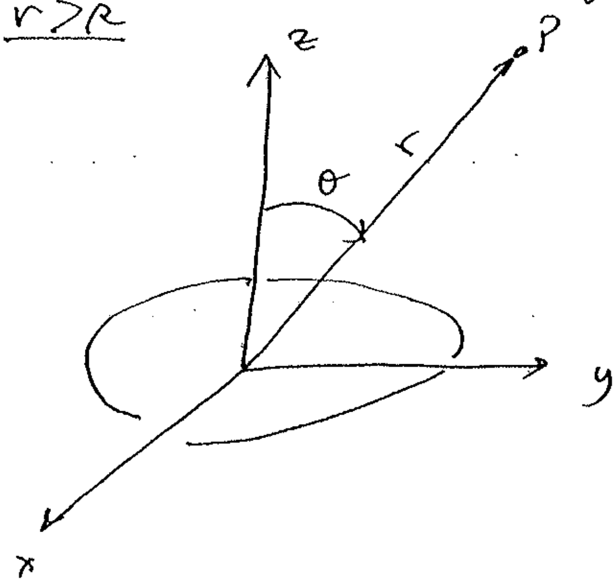
$$\Phi = \left[\Phi_R - \frac{Q}{4\pi\epsilon_0 R} - E_0 r \mu\right] + \frac{Q}{4\pi\epsilon_0 r} + \frac{E_0 R^3}{r^2}\mu$$

if we gauge the Φ of the charged sphere to 0 at ∞, $\Phi_R = (Q/4\pi\epsilon_0 R)$ and

$$\boxed{\Phi = \left(\frac{R^3}{r^3} - 1\right) E_0 r \mu + \frac{Q}{4\pi\epsilon_0 r}}$$

Prob 3.21

a) Find the $V(r,\theta)$ for a charged disk off axis. Assume $r > R$



Use:

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\theta)$$

and $\theta = 0$

$$V(r,\theta) = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right]$$

to find the lowest order terms.

Solution:

(i) note: $P_l(y=1) = 1$, where $y = \cos\theta$

$$\Rightarrow V(r,\theta) = V(r,\theta=0) = V(r, y=1) \text{ and so,}$$

$$V(r,0) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\theta=0) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}}$$

(ii) we also have

$$V(r,\theta=0) = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right]$$

$$= \frac{\sigma}{2\epsilon_0}\left[r\sqrt{1+\left(\frac{R}{r}\right)^2} - r\right]$$

$$\approx \frac{\sigma}{2\epsilon_0}\left[r\left(1+\frac{1}{2}\frac{R^2}{r^2} - \frac{1}{8}\frac{R^4}{r^4} + \frac{1}{16}\frac{R^6}{r^6}\right) - r\right]$$

$$= \frac{\sigma}{2\epsilon_0}\left[\frac{1}{2}\frac{R^2}{r} - \frac{1}{8}\frac{R^4}{r^3} + \frac{1}{16}\frac{R^6}{r^5}\right]$$

$$= \frac{\sigma R^2}{4r\epsilon_0}\left[1 - \frac{1}{4}\frac{R^2}{r^2} + \frac{1}{8}\frac{R^4}{r^4}\right]$$

$$\Rightarrow \quad B_0 = \frac{\sigma R^2}{4\epsilon_0}$$

$$B_2 = -\frac{\sigma R^4}{16\epsilon_0}$$

$$B_4 = \frac{\sigma R^6}{32\epsilon_0}$$

$$\vdots$$

and $B_\ell = 0$, ℓ odd

so that

$$V(r,\theta) \approx \sum_{\ell\,\text{even}}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\theta)$$

b) Find $V(r,\theta)$ for $r < R$

(i) given, $V(r,\theta=0) = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right]$

$$\approx \frac{\sigma R}{2\epsilon_0}\left[1 + \frac{1}{2}\frac{r^2}{R^2} - \frac{1}{8}\frac{r^4}{R^4} + \frac{1}{16}\frac{r^6}{R^6} + \dots - \frac{r}{R}\right]$$

(ii) $V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\theta)$

let $\theta = 0 \Rightarrow P_\ell(1) = 1$

above xy plane

$$\begin{cases} A_0^> = \frac{\sigma R}{2\epsilon_0} \\ A_1^> = -\frac{\sigma}{2\epsilon_0} \\ A_2^> = \frac{\sigma R}{4\epsilon_0 R^2} = \frac{\sigma}{4\epsilon_0 R} \\ A_3^> = 0 \end{cases}$$

$$A_4^> = -\frac{\sigma}{16\epsilon_0 R^3}$$

$$\vdots$$

let $\theta = \pi \rightarrow P_\ell(-1) = (-1)^\ell$

below xy plane

$$\left\{ \begin{array}{ll} A_0^< = \dfrac{\sigma R}{2\epsilon_0} & A_4^< = -\dfrac{\sigma}{16\epsilon_0 R^3} \\ A_1^< = \dfrac{\sigma}{2\epsilon_0} & \vdots \\ A_2^< = \dfrac{\sigma}{4\epsilon_0 R} & \\ A_3^< = 0 & \end{array} \right.$$

Sol^ns are

$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty} \begin{pmatrix} A_\ell^> \\ A_\ell^< \end{pmatrix} r^\ell P_\ell(\theta)}$$

Prob 3.23

Solve Laplace's Eqn. by separation of Variables

$$\nabla^2 V = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right] V = 0$$

let $V = R(r)\Phi(\phi)$

$$\nabla^2 V = \left\{\frac{\Phi(\phi)}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} R(r) + \frac{R(r)}{r^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}\right] = 0$$

$$\Rightarrow \frac{1}{R(r)r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + \frac{1}{\Phi(\phi) r^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} = 0$$

multiply by r^2

$$\underbrace{\frac{r}{R(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)}_{+m^2} + \underbrace{\frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}}_{-m^2} = 0$$

(A) and $\frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} = -m^2 \Rightarrow \frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + m^2 \Phi(\phi) = 0$

$$\boxed{\Phi(\phi) = A\cos m\phi + B\sin m\phi}$$

(B) $\frac{r}{R(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) = +m^2$

let $R(r) = C r^\alpha$

$$\frac{r}{R(r)}\frac{\partial}{\partial r}\left[\alpha C r^\alpha\right] = m^2$$

$$C\alpha^2 r^\alpha = m^2 C r^\alpha \Rightarrow \alpha^2 = m^2 \rightarrow \alpha = \pm m$$

and $\boxed{R(r) = C r^m + D r^{-m}}$

(C) <u>Consider case when $m=0$</u>

(i) $\dfrac{\partial^2 \Phi(\phi)}{\partial \phi^2} = 0 \Rightarrow \boxed{\Phi(\phi) = a\phi + b}$

(ii) $\dfrac{r}{R(r)} \dfrac{\partial}{\partial r}\left(r \dfrac{\partial R(r)}{\partial r} \right) = 0 \Rightarrow r \dfrac{\partial R(r)}{\partial r} = c$

$$\boxed{R(r) = c \ln r + d}$$

(D) Total solution is

$$\boxed{V(r,\phi) = \cancel{[c\ln r + d] + Cr^m + Dr^{-m}} \quad (c\ln r + d)(a\phi + b) + (Cr^m + Dr^{-m})(A\cos m\phi + B\sin m\phi)}$$

(E) General Solution is then

$$V(r,\phi) = (a\phi + b)(c\ln r + d) + \sum_{m=1}^{\infty}\left(C_m r^m + \frac{D_m}{r^m}\right)\left(A_m \cos m\phi + B_m \sin m\phi\right)$$