Physics 412: Introduction to Electrodynamics
Homework 7
Due: Friday, 2012 November 16

40. Problem 3.26

41. Problem 3.30

42. Problem 3.31

43. Problem 3.38

44. Problem 3.40

45. Problem 3.41

46. Problem 3.45

Prob 3.26

A sphere of radius R, centered at the origin, carries charge density

$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)\sin\theta$$

where k is a constant. Find the approximate $V(r,\theta)$ for points on the z-axis far from the sphere.

Sol^{n}

$$V(r,\theta{=}0) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int (r')^{l} P_l(u')\,\rho(\vec{r}')\,d\tau'$$

So, now, let's look at the "moments" which are nonzero.

$l=0$

$$= \int P_0(u')\,\rho(\vec{r}')\,d\tau'$$

$$= \int k\frac{R}{r'^2}(R-2r')\sin\theta'\sin\theta' d\theta' d\phi' dr' r'^2$$

$$= 2\pi k R\int_0^{\pi}\left(\frac{R-2r'}{r'^2}r'^2\right)\sin^2\theta' d\theta' dr'$$

$$= \pi k R\int (R-2r')dr'$$

$$= \pi k R\,(Rr'-r'^2)\Big|_0^R$$

$$= \pi k R(R^2-R^2)$$

$$= 0\,!$$

$l=1$

$$= \int r' P_1(u')\,k\frac{R}{r'^2}(R-2r')\sin\theta' r'^2 dr' \sin\theta' d\theta' d\phi'$$

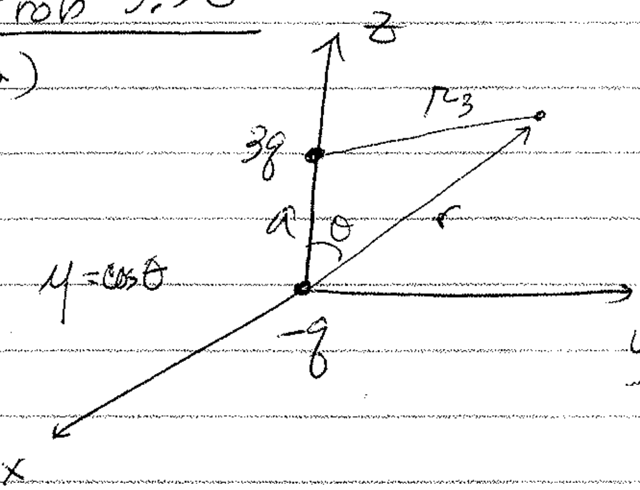
$$= 2\pi k R\int\!\!\int\left((R-2r')r'dr'\right)\cos\theta'\sin\theta'\sin\theta' d\theta'$$

$$= 2\pi k R\int r'(R-2r')dr'\int_0^0 \sin^2\theta' d(\sin\theta')$$

$$= 0$$

Prob 3.30

a)



i) Monopole $= \sum q_i = 2q$

ii) Dipole $= \sum q_i \vec{r}_i = 3qa\hat{z}$

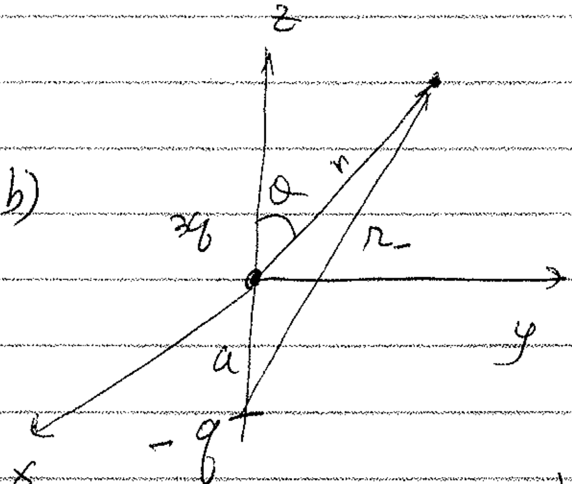
$$V(r,\mu) = \frac{1}{4\pi\epsilon_0}\left[\frac{3q}{\sqrt{a^2+r^2-2ar\cos\theta}} - \frac{q}{r}\right]$$

$$= \frac{q}{4\pi\epsilon_0 r}\left[3\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^l P_l(\cos\theta) - 1\right]$$

$$\cong \frac{q}{4\pi\epsilon_0 r}\left[3+3\left(\frac{a}{r}\right)\cos\theta - 1\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{2}{r} + 3\frac{a}{r^2}\cos\theta\right] \checkmark$$

b)



i) Monopole $= \sum q_i = 2q$

ii) Dipole $= \sum q_i \vec{r}_i = -qa(-\hat{z}) = qa\hat{z}$

$$V(r,\mu) = \frac{1}{4\pi\epsilon_0}\left[\frac{3q}{r} - \frac{q}{\sqrt{a^2+r^2-2ar\cos(\pi-\theta)}}\right]$$

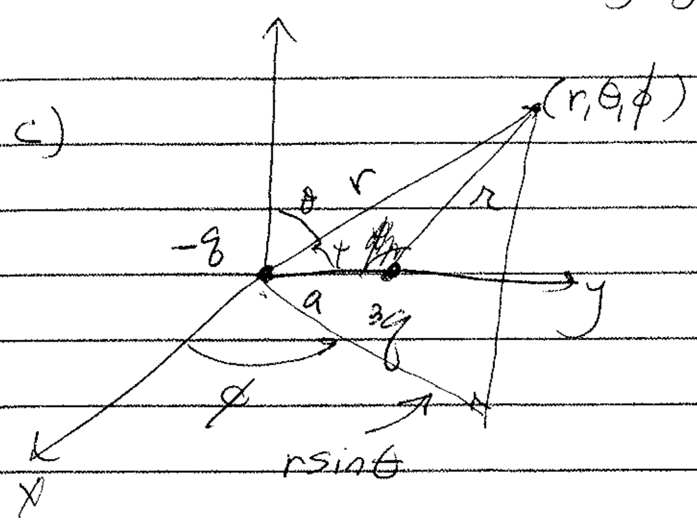
$$= \frac{q}{4\pi\epsilon_0 r}\left[3 - \sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^l P_l(-\cos\theta)\right]$$

$$= \frac{q}{4\pi\epsilon_0 r}\left[3 - 1 + \left(\frac{a}{r}\right)\cos\theta\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{2}{r} + \frac{a}{r^2}\cos\theta\right] \checkmark$$

not axially symmetric about z-axis

c)



i) Monopole $= \sum q_i = 2q$

ii) dipole $= \sum q_i \vec{r}_i = 3qa\,\hat{y}$

$$V(r,\psi) = \frac{1}{4\pi\epsilon_0}\left[\frac{-q}{r} + \frac{3q}{\sqrt{a^2+r^2-2ar\cos\psi}}\right]$$

note: $\cos\psi = \hat{r}\cdot\hat{a} = \dfrac{(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\cdot(0,a,0)}{ra}$

$= \sin\theta\sin\phi$

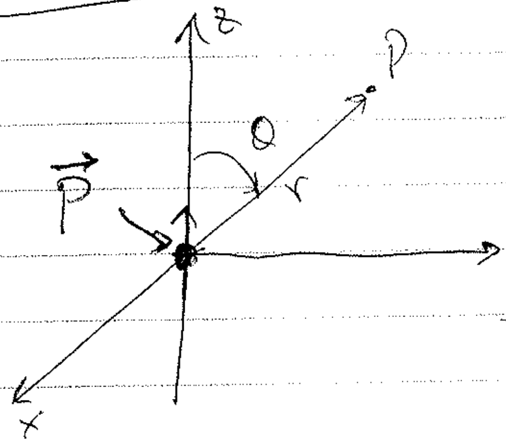
$$\Rightarrow V(r,\psi) = \frac{1}{4\pi\epsilon_0}\left[-\frac{q}{r} + \frac{3q}{\sqrt{a^2+r^2-2ar\sin\theta\sin\phi}}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[-\frac{q}{r} + \frac{3q}{r}\sum_{\ell=0}^{\infty}\left(\frac{a}{r}\right)^{\ell} P_\ell(\cos\psi)\right]$$

$$\simeq \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3q}{r}\left(\frac{a}{r}\right)P_1(\cos\psi)\right]$$

$$\simeq \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa}{r^2}\sin\theta\sin\phi\right]$$ ✓

Problem 3.31



a) What is the force on a point charge at $(a,0,0)$?

$$V_d = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{P}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{P}{r^2}\cos\theta$$

$$\Rightarrow \vec{E}_d = \frac{P}{4\pi\epsilon_0}\left[2\frac{\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right]$$

at $x=a$ $(\theta = \pi/2)$ $(r=a)$

$$\vec{E}_d = \frac{P}{4\pi\epsilon_0}\left[0 + \frac{1}{a^3}\hat{\theta}\right]$$

$$\Rightarrow \vec{F} = q\vec{E}_d = \frac{P}{4\pi\epsilon_0 a_0^2}\hat{\theta}$$

$$\boxed{\vec{F} = \frac{P}{4\pi\epsilon_0 a_0^2}(-\hat{z})}$$

b) what is the force on a charge q at $(0,0,a)$?

$$\vec{E}_d = \frac{P}{4\pi\epsilon_0}\left[\frac{2}{a^3}\hat{r}\right], \quad (0,0,a)\rightarrow \theta = 0,\ r=a$$

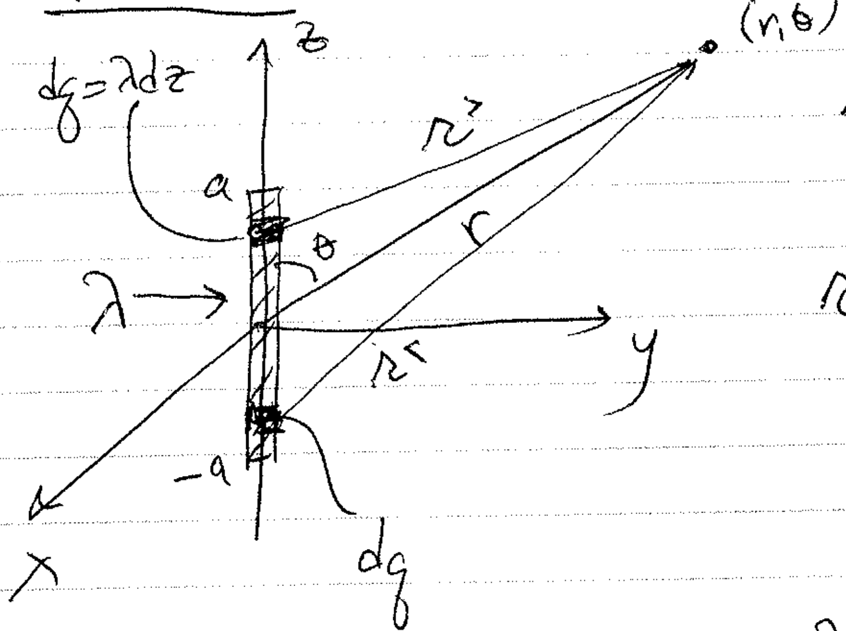
$$\Rightarrow \boxed{\vec{F} = q\vec{E}_d = \frac{P}{2\pi\epsilon_0 a^3}\hat{z}}$$

c) How much work does it take to move q from $(a,0,0)$ to $(0,0,a)$?

$$\text{Work} = +q\int_A^B dV = q\left[\frac{1}{4\pi\epsilon_0}\frac{P}{a^2} - \frac{1}{4\pi\epsilon_0}\times 0\right]$$

$$\boxed{\text{Work} = \frac{1}{4\pi\epsilon_0}\frac{qP}{a^2}}$$

Prob 3.38



$$r^> = \sqrt{z^2 + r^2 - 2zr\cos\theta}$$

$$r^< = \sqrt{z^2 + r^2 - 2r(-z)\cos(\pi - \theta)}$$

we want the length to be positive

enclosed ∡ subtended at $z = 0$

$$\Rightarrow dV = \frac{\lambda}{4\pi\epsilon_0} \frac{dz}{\sqrt{z^2 + r^2 - 2rz\cos\theta}}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_{-a}^{a} \frac{dz}{\sqrt{z^2 + r^2 - 2rz\cos\theta}}$$

for $r > a$

$$= \frac{\lambda}{4\pi\epsilon_0 r} \int_{-a}^{a} \frac{dz}{\sqrt{1 + \left(\frac{z}{r}\right)^2 - 2\left(\frac{z}{r}\right)\cos\theta}}$$

$$= \frac{\lambda}{4\pi\epsilon_0 r} \int_{-a}^{a} dz \sum_{l=0}^{\infty} \left(\frac{z}{r}\right)^l P_l(\cos\theta)$$

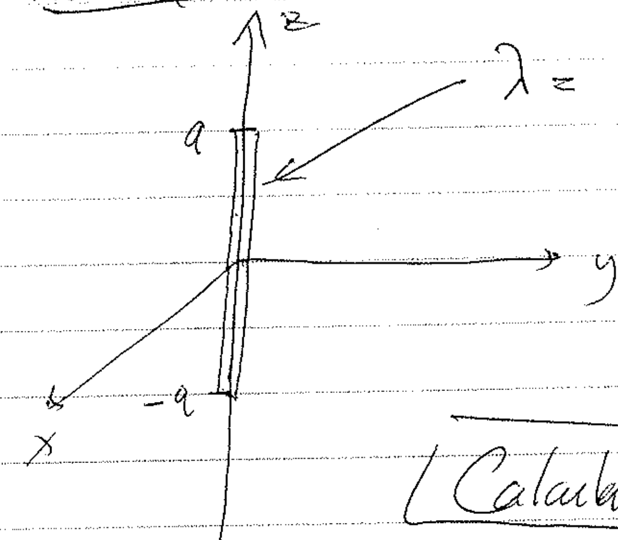
$$= \frac{\lambda}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \left(\frac{1}{r}\right)^{l+1} \int_{-a}^{a} \left\{ P_0(\cos\theta) + \cancel{z P_1(\cos\theta)}^{\,0} + z^2 P_2(\cos\theta) + \dots \right\} dz$$

as will all other terms odd in l

$$= \frac{\lambda}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \left(\frac{1}{r}\right)^{l+1} \left[\frac{z^{l+1}}{l+1} P_l(\cos\theta) \right]_{-a}^{a} \text{ for } l \text{ even}$$

$$\boxed{V(r,\theta) = \frac{\lambda}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \left(\frac{1}{r}\right)^{\ell+1} \left[\frac{2a^{\ell+1}}{\ell+1} P_\ell(\cos\theta) \right]}$$

Problem 3.40



$$\lambda = \begin{cases} k\cos\left(\frac{\pi z}{2a}\right) & A \\ k\sin\left(\frac{\pi z}{a}\right) & B \\ k\cos\left(\frac{\pi z}{a}\right) & C \end{cases}$$

for the above λ, find V (the leading term in $V(r)$)

Calculate moments

(A) a) $Q = \int_{-a}^{a} \lambda\, dz = k\int_{-a}^{a} \cos\left(\frac{\pi z}{2a}\right) dz$

$$= +\frac{2ak}{\pi}\sin\left(\frac{\pi z}{2a}\right)\Bigg|_{-a}^{a}$$

$$= +\frac{2ak}{\pi}\left[1-(-1)\right] = +\frac{4ak}{\pi}$$

b) $\vec{p} = \int_{-a}^{a} \lambda \vec{z}\, dz = k\hat{z}\int_{-a}^{a} z\cos\left(\frac{\pi z}{2a}\right) dz$

set $u = z$, $dV = \cos\left(\frac{\pi z}{2a}\right) dz$

$du = dz$, $V = \frac{+2a}{\pi}\sin\left(\frac{\pi z}{2a}\right)$

$$\Rightarrow \vec{p} = \hat{z}k\left[\frac{2az}{\pi}\sin\left(\frac{\pi z}{2a}\right)\Bigg|_{-a}^{a} - \frac{2a}{\pi}\int_{-a}^{a}\sin\left(\frac{\pi z}{2a}\right)dz\right]$$

$$= 0 + \frac{4a^2}{\pi^2}\cos\left(\frac{\pi z}{2a}\right)\Bigg|_{-a}^{a}\hat{z}$$

$$\vec{p} = 0$$

$$\Rightarrow \boxed{V(r,\psi) = +\frac{4ak}{\pi}\frac{1}{4\pi\epsilon_0 r} = \frac{ak}{\pi^2\epsilon_0 r}}$$

(B) a) $Q = \int_{-a}^{a} k \sin\left(\frac{\pi z}{a}\right) dz$

$= -\frac{ka}{\pi} \cos\left(\frac{\pi z}{a}\right)\Big|_{-a}^{a} = -\frac{ka}{\pi}\cancel{\left[-1-(-1)\right]} \to 0$

b) $\vec{p} = \int_{-a}^{a} k \sin\left(\frac{\pi z}{a}\right) \vec{z}\, dz$

$= k\hat{z} \int_{-a}^{a} z \sin\left(\frac{\pi z}{a}\right) dz$

let $u = z$, $dV = \sin\left(\frac{\pi z}{a}\right) dz$

$du = dz$ $\quad V = -\frac{a}{\pi}\cos\left(\frac{\pi z}{a}\right)$

$\vec{p} = -\frac{ak}{\pi} z \cos\left(\frac{\pi z}{a}\right)\Big|_{-a}^{a} \hat{z} + k\frac{a}{\pi} \int_{-a}^{a} \hat{z} \cos\left(\frac{\pi z}{a}\right) dz$

$= -\frac{ak}{\pi}\left[-a-(+a)\right]\hat{z} + k\frac{a^2}{\pi^2} \cancel{\sin\left(\frac{\pi z}{a}\right)\Big|_{-a}^{a}}$

$\vec{p} = \frac{2a^2k}{\pi}\hat{z}$

$\Longrightarrow V(r,\eta) = \frac{1}{4\pi\epsilon_0} \frac{2a^2k}{\pi r^2} \eta$

$$\boxed{V(r,\eta) = \frac{a^2k\eta}{2\pi^2\epsilon_0 r^2}}$$

(C) a) $Q = \int_{-a}^{a} k\cos\left(\frac{\pi z}{a}\right)dz$

$= \frac{ak}{\pi}\sin\left(\frac{\pi z}{a}\right)\Big|_{-a}^{a}$

$= 0$

b) $\vec{p} = \hat{z}\int_{-a}^{a} kz\cos\left(\frac{\pi z}{a}\right)dz$

$= \hat{z}k\int_{-a}^{a} z\cos\left(\frac{\pi z}{a}\right)dz$

let $u = z$, $dV = \cos\left(\frac{\pi z}{a}\right)dz$

$du = dz \quad V = \frac{a}{\pi}\sin\left(\frac{\pi z}{a}\right)$

$\vec{p} = \hat{z}k\left[\cancel{\frac{a}{\pi}z\sin\left(\frac{\pi z}{a}\right)}^{0}\Big|_{-a}^{a} - \frac{ka\hat{z}}{\pi}\int_{-a}^{a}\sin\left(\frac{\pi z}{a}\right)dz\right.$

$= \frac{ak}{\pi}\frac{a}{\pi}\cos\left(\frac{\pi z}{a}\right)\Big|_{-a}^{a}$

$= \frac{ka^2}{\pi^2}\left[\cancel{-1-(-1)}^{0}\right]$

c) $Q_u = \int_{-a}^{a} z^2 P_2(1)\,\rho(z)\,dz$

$= \int_{-a}^{a} z^2 k\cos\left(\frac{\pi z}{a}\right)dz$

let $u = z^2$, $dV = \cos\left(\frac{\pi z}{a}\right)dz$

$du = 2z\,dz, \quad V = \frac{a}{\pi}\sin\left(\frac{\pi z}{a}\right)$

$Q_u = \cancel{\frac{a}{\pi}z^2\sin\left(\frac{\pi z}{a}\right)}^{0}\Big|_{-a}^{a} - \frac{2a}{\pi}\int_{-a}^{a} z\sin\left(\frac{\pi z}{a}\right)dz$

$$Q_\mu = -\frac{2a}{\pi}\int_{-a}^{a} z \sin\left(\frac{\pi z}{a}\right) dz$$

Let: $u = z$, $dV = \sin\left(\frac{\pi z}{a}\right) dz$

$du = dz$ $\quad V = -\frac{a}{\pi}\cos\left(\frac{\pi z}{a}\right)$

$$\Rightarrow Q_\mu = -\frac{2a}{\pi}\left[-\frac{az}{\pi}\cos\left(\frac{\pi z}{a}\right)\Big|_{-a}^{a} + \int_{-a}^{a}\frac{a}{\pi}\cos\left(\frac{\pi z}{a}\right) dz\right]$$

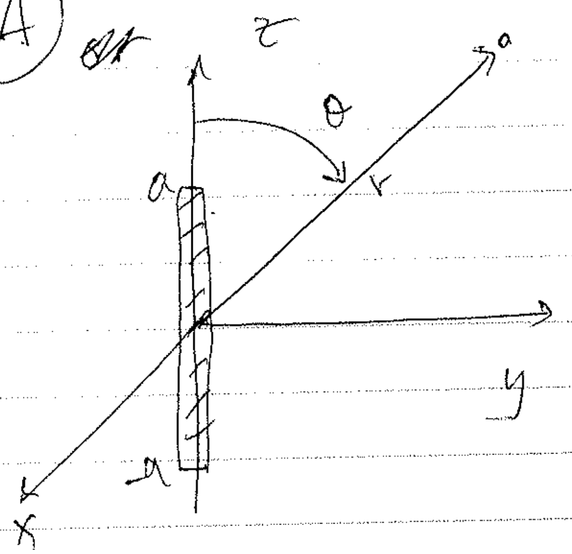
$$= -\frac{2a}{\pi}\left[+\frac{a^2}{\pi} - \left(-\frac{a^2}{\pi}\right)\right] - \frac{2a^2}{\pi^2}\int_{-a}^{a}\cos\left(\frac{\pi z}{a}\right) dz$$

$$= -\frac{4a^3}{\pi^2} - \frac{2a^2}{\pi^2}\frac{a}{\pi}\cancel{\sin\left(\frac{\pi z}{a}\right)\Big|_{-a}^{a}}$$

$$\Rightarrow \boxed{V(r,\mu) = \frac{-1}{\pi\epsilon_0}\left(\frac{a^3}{\pi^2 r^3}\right)P_2(\mu)}$$

alternatively, just solve for potential

(A)



$$V(r,\psi) = \frac{1}{4\pi\epsilon_0}\int_{-a}^{a} \frac{k\cos\left(\frac{\pi z}{2a}\right)dz}{\sqrt{r^2+z^2-2rz\cos\theta}}$$

$$\overset{r \gg a}{\approx} \frac{k}{4\pi\epsilon_0 r}\int_{-a}^{a} \frac{\cos\left(\frac{\pi z}{2a}\right)dz}{\sqrt{1+\frac{z^2}{r^2}-2\frac{z}{r}\cos\theta}}$$

$$= \frac{k}{4\pi\epsilon_0 r}\int_{-a}^{a}\cos\left(\frac{\pi z}{2a}\right)\left[\sum_{l=0}^{\infty}\left(\frac{z}{r}\right)^{l}P_l(\cos\theta)\right]dz$$

$$V(r,\psi) = \sum_{l=0}\left(\frac{k P_l(\cos\theta)}{4\pi\epsilon_0 r^{l+1}}\int_{-a}^{a} z^l\cos\left(\frac{\pi z}{2a}\right)dz\right)$$

(i) $l=0$; $\int_{-a}^{a}\cos\left(\frac{\pi z}{2a}\right)dz = +\frac{2a}{\pi}\sin\left(\frac{\pi z}{2a}\right)\Big|_{-a}^{a} = \frac{4a}{\pi}$

$$\Rightarrow \boxed{V(r,\psi) = \frac{ka}{\pi^2\epsilon_0 r}}$$

(B) $V(r,\psi) = \frac{1}{4\pi\epsilon_0}\int_{-a}^{a}\frac{k\sin\left(\frac{\pi z}{a}\right)dz}{\sqrt{r^2+z^2-2rz\cos\theta}}$

$$= \frac{k}{4\pi\epsilon_0 r}\sum_{l=0}^{\infty}\int_{-a}^{a}\sin\left(\frac{\pi z}{a}\right)\left(\frac{z}{r}\right)^{l}P_l(\cos\theta)$$

(i) $l=0$; $\int_{-a}^{a}\sin\left(\frac{\pi z}{a}\right)dz = -\frac{a}{\pi}\cancel{\cos\left(\frac{\pi z}{a}\right)}\Big|_{-a}^{a} \to 0$

(ii) $l=1$ $\int_{-a}^{a}\frac{z}{r}\sin\left(\frac{\pi z}{a}\right)P_1(\cos\theta)dz$

$$= \frac{P_1(\cos\theta)}{r}\int_{-a}^{a} z\sin\left(\frac{\pi z}{a}\right)dz$$

$$= \frac{2a^2}{\pi r}P_1(\cos\theta)$$

$$\Rightarrow \boxed{V(r,\theta) = \frac{ka^2}{2\pi^2 \epsilon_0 r^2} P_1(\cos\theta)}$$

(c) $V(r,\theta) = \frac{1}{4\pi\epsilon_0}\int_{-a}^{a} \frac{k\cos(\frac{\pi z}{a})\,dz}{\sqrt{r^2+z^2-2rz\cos\theta}}$

$$= \frac{k}{4\pi\epsilon_0 r}\sum_{l=0}^{\infty}\frac{P_l(\cos\theta)}{r^l}\int_{-a}^{a}\cos(\pi z/a)\,dz\, z^l$$

$$= \sum_{l=0}^{\infty}\left[\frac{kP_l(\cos\theta)}{4\pi\epsilon_0 r^{l+1}}\int_{-a}^{a} z^l\cos\left(\frac{\pi z}{a}\right)dz\right]$$

(i) $l=0$; $\int_{-a}^{a}\cos\left(\frac{\pi z}{a}\right)dz = \frac{a}{\pi}\cancelto{0}{\sin\left(\frac{\pi z}{a}\right)}\Big|_{-a}^{a}$

(ii) $l=1$; $\int_{-a}^{a} z\cos\left(\frac{\pi z}{a}\right)dz$

$$= \left(\frac{a}{\pi}\right)^2\int_{-\pi}^{\pi} x\cos x\,dx$$

$$= \left(\frac{a}{\pi}\right)^2\left[\cancelto{0}{\cos x + x\sin x}\right]_{-\pi}^{\pi}$$

(iii) $l=3$; $\int_{-a}^{a} z^2\cos\left(\frac{\pi z}{a}\right)dz$

$$= \left(\frac{a}{\pi}\right)^3\int_{-\pi}^{\pi}(x^2\cos x)\,dx$$

$$= \left(\frac{a}{\pi}\right)^3\left[\underbrace{2x\cos x}_{-2\pi-(+2\pi)} + \cancelto{0}{(x^2-2)\sin x}\right]_{-\pi}^{\pi}$$

$$= -4\pi\left(\frac{a^3}{\pi^3}\right) = -\frac{4a^3}{\pi^2}$$

$$\Rightarrow V(r_{in}) = \frac{k P_2(\cos\theta)}{4\pi\epsilon_0 r^3}\left[-\frac{4a^3}{\pi^2}\right]$$

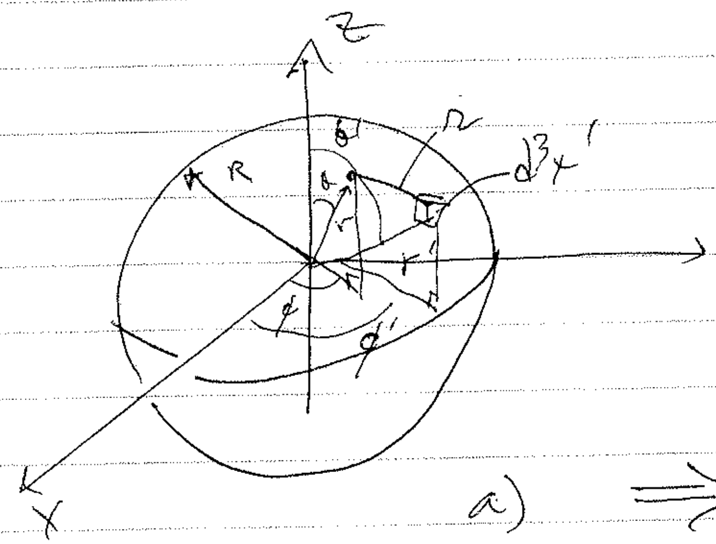
$$= -\frac{k a^3}{\cancel{4}\pi^3 \epsilon_0 r^3} P_2(\cos\theta)$$

Problem 3.41

ⓐ Show that the average field due to a single charge q at point $\vec{r}$ inside the sphere is the same as the field at $\vec{r}$ due to an uniformly charged sphere w/ $\rho = -q/(\frac{4\pi}{3}R^3)$, namely

$$\frac{1}{4\pi\epsilon_0}\frac{q}{\left(\frac{4\pi}{3}R^3\right)}\int\frac{\hat{r}\,d^3x'}{\mathcal{R}^2}$$

where $\vec{\mathcal{R}}$ is the vector from $\vec{r}$ to d^3x'



the solution of the integral can be simplified by noting that if we place the charge q on the z-axis, we do not lose any information. Set $\theta = \phi = 0$

a) $\Rightarrow \mathcal{R}^2 = r^2 + r'^2 - 2rr'\cos\theta'$

b) $\Rightarrow$ that all "horizontal" components vanish and we need only keep the z-compont of the field

$$\Rightarrow dE_z = \hat{z}\cdot d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q\,d^3x'}{(r^2+r'^2-2rr'\cos\theta')}\frac{r-r'\cos\theta'}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}$$

$$\int dE_z = \frac{-q}{4\pi\epsilon_0}\int\frac{(r-r'\cos\theta')}{(r^2+r'^2-2rr'\cos\theta')^{3/2}}\,r'^2\sin\theta'\,d\theta'\,d\phi'\,dr'$$

$$E_z = \frac{-q}{4\pi\epsilon_0}\underbrace{\int \frac{(r - r'\cos\theta')}{(r^2 + r'^2 - 2rr'\cos\theta')^{3/2}} r'^2 dr' \sin\theta' d\theta' d\phi'}$$

we have seen this integral before. Let's do it again, anyway

$$= \frac{-q}{2\epsilon_0}\int_0^R\int_0^\pi \frac{(r - r'\cos\theta') r'^2 dr' \sin\theta' d\theta'}{(r^2 + r'^2 - 2rr'\cos\theta')^{3/2}}$$

Comment: $\cancel{\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}\left[a + bx - a\ln(a+bx)\right]}$

$$\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}} \ \& \int \frac{dx}{X\sqrt{X}} = \frac{2(2cx+b)}{q\sqrt{X}}$$

where $X = a + bx + cx^2$.

So that if, $x = \cos\theta'$

$$\left.\begin{aligned} a &= r^2 + r'^2 \\ b &= -2rr' \\ c &= 0 \end{aligned}\right\} q = 4ac - b^2$$

the $\sin\theta' d\theta' = d(-\cos\theta')$ integration becomes

$$E_z = \frac{+q}{2\epsilon_0}\int_0^R r'^2 dr' \left\{ r\left[\frac{2(-2rr')}{-4r^2r'^2\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}}\right] - r'\left[\frac{-2(-2rr'\cos\theta' + 2(r^2+r'^2))}{-(-2rr')^2\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}}\right]\right\}\Bigg|_0^\pi$$

$$E_z = \frac{+q}{2\epsilon_0}\int_0^R r'^2 dr' \left\{ \frac{+\cancel{x}}{\cancel{x} r'}\left[\frac{1}{(r+r')} - \frac{1}{|r-r'|}\right] \right.$$

$$+ \frac{\cancel{x}'}{r\cancel{x}}\left[\frac{-1}{(r+r')} - \frac{1}{|r-r'|}\right]$$

$$\left. + \frac{\cancel{x}'(r^2+r'^2)}{r^2 r' \cancel{x}}\left[\frac{1}{(r+r')} - \frac{1}{|r-r'|}\right]\right\}$$

look at $r'>r$ part of integral

$$E_z^{r'>r} = +\frac{q}{2\epsilon_0}\int_r^R dr' \left\{ +r'\left(\frac{2r}{r^2-r'^2}\right) + \frac{r'^2}{r}\left(\frac{+2r'}{r^2-r'^2}\right) \right.$$

$$\left. + \frac{r'(r^2+r'^2)}{r^2}\left(\frac{2r}{r^2-r'^2}\right)\right\}$$

$$= \frac{+q}{2\epsilon_0}\int_r^R dr' \left\{ \frac{+2rr' + 2\frac{r'^3}{r} + 2\frac{r'}{r}(r^2+r'^2)}{(r^2-r'^2)} \right\}$$

$$= \frac{+q}{\epsilon_0}\int_r^R dr' \left\{ \frac{+rr' + \frac{r'^3}{r} + rr' + \frac{r'^3}{r}}{(r^2-r'^2)} \right\}$$

$$\boxed{E_z^{r'>r} = 0}$$

look at $r'<r$ part of integral

$$E_z^{r'<r} = \frac{+q}{2\epsilon_0}\int_0^r r'^2 dr' \left\{ \frac{1}{r'}\left[\frac{1}{r+r'} - \frac{1}{r-r'}\right] + \frac{1}{r}\left[-\frac{1}{r+r'} - \frac{1}{r-r'}\right] - \frac{(r^2+r'^2)}{r^2 r'}\left[\frac{1}{r+r'} - \frac{1}{r-r'}\right] \right\}$$

$$= \frac{+q}{2\epsilon_0}\int_0^r r'^2 dr' \left\{ \frac{1}{r'}\left(\frac{-2r'}{r^2-r'^2}\right) - \frac{1}{r}\left(\frac{2r}{r^2-r'^2}\right) - \left(\frac{r^2+r'^2}{r^2 r'}\right)\left(\frac{-2r'}{r^2-r'^2}\right) \right\}$$

$$= \frac{+q}{2\epsilon_0}\int_0^r r'^2 dr' \left[\frac{-\cancel{2} - \cancel{2} + 2\left(\frac{r^2+r'^2}{r^2}\right)}{(r^2-r'^2)} \right]$$

$$= \frac{+q}{\epsilon_0 r^2}\int_0^r r'^2 dr \left[\frac{(r^2+r'^2)}{(r^2-r'^2)} - \frac{2r^2}{(r^2-r'^2)} \right]$$

$$= \frac{+q}{\epsilon_0 r^2}\int_0^r r'^2 dr' \left[\frac{r'^2 - r^2}{(r^2-r'^2)} \right]$$

$$= -\frac{q}{\epsilon_0 r^2}\int_0^r r'^2 dr'$$

$$\boxed{E_z^{r'<r} = -\frac{q}{\epsilon_0 r^2}\left(\frac{r'^3}{3}\right)\Big|_0^r = -\frac{qr}{3\epsilon_0}}$$

The average field over the volume, $\frac{4\pi}{3}R^3$ is then

$$\boxed{\overline{E}_z = \frac{E_z^{r'<r}}{\frac{4\pi}{3}R^3} = -\frac{q r}{3\epsilon_0\left(\frac{4\pi}{3}R^3\right)}\hat{z}} \Rightarrow \boxed{\overline{E}_r = -\frac{q\vec{r}}{4\pi\epsilon_0 R^3}}$$

(b) Dipole moment of q is $q\vec{r}$ (with respect to origin)

$$\Rightarrow \boxed{\overline{E}_z = -\frac{q\vec{r}}{3\epsilon_0\frac{4\pi}{3}R^3} = -\frac{\vec{p}}{4\pi\epsilon_0 R^3}}$$

(c) For arbitrary charge distribution

$$\boxed{\overline{E}_{z,total} = \sum \overline{E}_{z_i} = -\frac{\sum \vec{p}_i}{4\pi\epsilon_0 R^3}}$$

$\vec{P}$ = total vector sum of $\vec{p}$'s.

(d) for charges outside volume use the $E_z^{r'<r}$ result

$$E_z^{r'<r} = -\frac{q}{\epsilon_0 r^2}\left(\frac{r'^3}{3}\right)\Big|_0^R = -\frac{qR^3}{3\epsilon_0 r^2}$$

for 1 charge

$$E_z^{r'<r} = \frac{4\pi R^3}{3}\left(-\frac{q}{4\pi\epsilon_0}\right)\frac{1}{r^2}$$ in direction of charge

$$\Rightarrow \boxed{\overline{E}_r = -\frac{4\pi R^3}{3}\sum_{i=1}^{N}\frac{q_i}{4\pi\epsilon_0 r_i^2}\hat{r}_i}$$

Problem 3.45

a) Show that the quadruple term in the multipole expansion can be written

$$V_{quad}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\sum_{i,j}^{3} \hat{r}_i \hat{r}_j Q_{ij}$$

where $Q_{ij} = \int\left[3r_i' r_j' - (r')^2\delta_{ij}\right]\rho(\vec{r}')\,d^3x'$

Solution

$$V_{quad} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r^3}\right)\int r'^2 P_2(\cos\theta')\rho(\vec{r}')\,d^3x'$$

$n=2$ term in Eqn (3.95) in text

$$= \frac{1}{4\pi\epsilon_0 r^3}\int r'^2\left[\frac{1}{2}(3\cos^2\theta' - 1)\rho(\vec{r}')\right]d^3x'$$

from §3.4.2 in text (and class notes),

$$r'\cos\theta' = \hat{r}\cdot\vec{r}' \Rightarrow \cos\theta' = \hat{r}\cdot\hat{r}'$$

$$V_{quad} = \frac{1}{4\pi\epsilon_0 r^3}\int \cancel{r'^2}\left[\frac{1}{2}\left(3\left[\frac{\hat{r}\cdot\hat{r}' r'}{\cancel{r'}}\right]^2 - \cancel{1}\right)\right]\rho(\vec{r}')\,d^3x'$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\int\left[\frac{1}{2}\left(3\left[\hat{r}\cdot\vec{r}'\right]^2 - r'^2\right]\rho(\vec{r}')\,d^3x'$$

$$\vec{r}' = \frac{\hat{r}' r'}{\cancel{r'}}$$

what is $\hat{r}\cdot\hat{r}'$?

$$\hat{r}\cdot\hat{r}' = (\hat{x} + \hat{y} + \hat{z})\cdot(\hat{x}' + \hat{y}' + \hat{z}')$$

$$= \frac{xx' + yy' + zz'}{rr'}$$

$$(\hat{r}\cdot\hat{r}')^2 = \frac{(xx')^2 + (xx')(yy') + (xx')(zz') + (yy')(xx') + \cdots}{r^2r'^2}$$

$$+ \frac{(yy')^2 + (yy')(zz') + (zz')(xx') + (zz')(yy') + (zz')^2}{r^2r'^2}$$

$$r^2r'^2(\hat{r}\cdot\hat{r}')^2 = x\left[(x)x'^2 + yx'y' + zx'z'\right] + y\left[xx'y' + yy'^2 + zy'z'\right]$$
$$+ z\left[xx'z' + yy'z' + zz'^2\right]$$

to make this look like the one in the text, let

$$\boxed{\begin{array}{l} x_i = x, y, z \text{ for } i = 1,2,3 \\ \text{and} \\ x_i' = x', y', z' \text{ for } i = 1,2,3 \end{array}}$$

$$\cancel{\Rightarrow r^2r'^2(\hat{r}\cdot\hat{r}')^2 = x_i x_i}$$

$$\Rightarrow r^2 r'^2 (\hat{r}\cdot\hat{r}')^2 = x_1 x_1 (x_1' x_1') + x_2 x_2 (x_2' x_2') + x_3 x_3 (x_3' x_3')$$
$$+ x_1 x_2 (x_1' x_2') + x_1 x_3 x_1' x_3' + x_1 x_2 (x_1' x_2')$$
$$+ x_2 x_3 x_2' x_3' + x_1 x_3 x_1' x_3' + x_2 x_3 x_2' x_3'$$

group terms like and so on ...

$$r^2 r'^2 (\hat{r}\cdot\hat{r}')^2 = x_1 \left[x_1 (x_1' x_1') + x_2 (x_1' x_2') + x_3 (x_1' x_3') \right]$$
$$+ x_2 \left[x_1 (x_2' x_1') + x_2 (x_2' x_2') + x_3 (x_2' x_3') \right]$$
$$+ x_3 \left[x_1 (x_1' x_3') + x_2 (x_2' x_3') + x_3 (x_3' x_3') \right]$$
$$= \sum_{i=1}^{3} x_i \left[\sum_{j=1}^{3} x_j \, (x_i' x_j') \right]$$
$$= \sum_{i=1}^{3} \sum_{j=1}^{3} x_i x_j (x_i' x_j')$$
$$(\hat{r}\cdot\hat{r}')^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \hat{x}_i \hat{x}_j (\hat{x}_i' \cdot \hat{x}_j') \quad \text{after dividing by } \underline{r^2 r'^2}$$

plug into Vquad formula

$$V_{quad} = \frac{1}{4\pi\epsilon_0 r^3}\int \frac{1}{2}\left(3\sum_{i=1}^{3}\sum_{j=1}^{3}\hat{x}_i\hat{x}_j\left(\hat{x}_i'\hat{x}_j' r'^2\right) - r'^2\right)\rho(\vec{r}\,')\,d^3x'$$

$$= \frac{1}{8\pi\epsilon_0 r^3}\int\left[3\sum_{i=1}^{3}\sum_{j=1}^{3}\hat{x}_i\hat{x}_j\left(x_i'x_j'\right) - r'^2\right]\rho(\vec{r}\,')\,d^3x'$$

pull r'^2 into the $\sum_i\sum_j$

$$= \frac{1}{8\pi\epsilon_0 r^3}\int\left[\sum_{i=1}^{3}\sum_{j=1}^{3}\hat{x}_i\hat{x}_j\left(3x_i'x_j' - \delta_{ij}r'^2\right)\right]\rho(\vec{r}\,')\,d^3x'$$

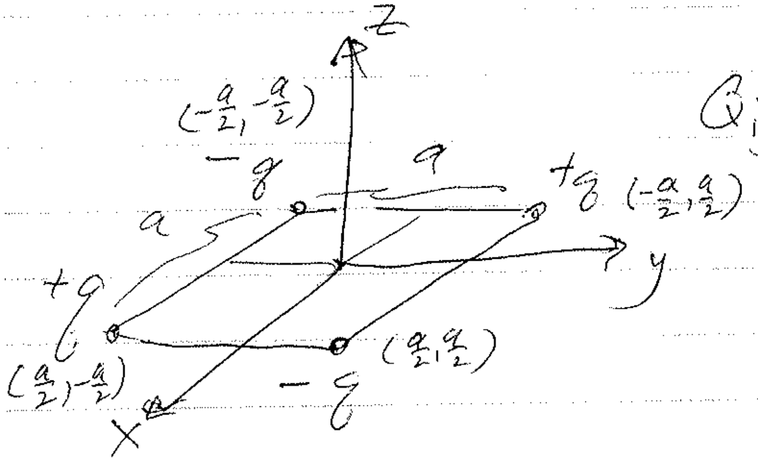
Q: the r'^2 comes in as it does ~~because~~ not as $\frac{3r'^2}{3}\delta_{ij}$...

A: what happened. We pulled into the term multiplied by $\hat{x}_i\hat{x}_j$ which $\hat{x}^2, \hat{y}^2, \hat{z}^2$ when $i=j \Rightarrow \hat{x}^2 = \frac{1}{3} = \hat{y}^2 = \hat{z}^2$ and so, the $\frac{1}{3}$ is "eaten up" by the $\hat{x}_i\hat{x}_j$ factor

Next, pull $\hat{x}_i\hat{x}_j$ out of the integral

$$\boxed{V_{quad} = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\sum_{i=1}^{3}\sum_{j=1}^{3}\hat{x}_i\hat{x}_j\underbrace{\int\left(3x_i'x_j' - r'^2\delta_{ij}\right)\rho(\vec{r}\,')\,d^3x'}_{Q_{ij}}}$$

b) Evaluate Q_{ij} for the configuration



$$Q_{ij} = \int \left[3x_i' x_j' - r'^2 \delta_{ij}\right] \rho(\vec{r}\,') \, d^3x'$$

$\rho(\vec{r}\,')$ is a δ-function, only nonzero at $(\frac{a}{2}, \frac{a}{2})$, $(-\frac{a}{2}, \frac{a}{2})$, $(-\frac{a}{2}, -\frac{a}{2})$, $(\frac{a}{2}, -\frac{a}{2})$

$\Rightarrow$ integral is easy.

<u>find Q_{ij}</u>

(i) $Q_{11} = \int \left[3x'x' - r'^2\right] \rho(\vec{r}\,') \, d^3x'$

$$= -q\left(3\frac{a^2}{4} - \frac{a^2}{2}\right) + q\left(3\frac{a^2}{4} - \frac{a^2}{2}\right) - q\left(3\frac{a^2}{4} - \frac{a^2}{2}\right) + q\left(3\frac{a^2}{4} - \frac{a^2}{2}\right)$$

$$= 0$$

(ii) $Q_{22} = \int \left[3y'y' - r'^2\right] \rho(\vec{r}\,') \, d^3x'$

$= 0$, by analogy to Q_{11}

(iii) $Q_{33} = 0$, because charges are in xy plane and Q_{ij} is traceless

(iv) $Q_{12} = \int \left[3x'y'\right] \rho(\vec{r}\,') \, d^3x'$

$$= -q\left[3\frac{a^2}{4}\right] + q\left[-3\frac{a^2}{4}\right] - q\left[3\frac{a^2}{4}\right] + q\left[-3\frac{a^2}{4}\right]$$

$$= -3qa^2$$

(v) $Q_{13} = 0$, no z ($z=0$ for all charges)

(vi) $Q_{21} = Q_{12} = -3qa^2$

(vii) $Q_{23} = 0$, $z=0$ for all charges

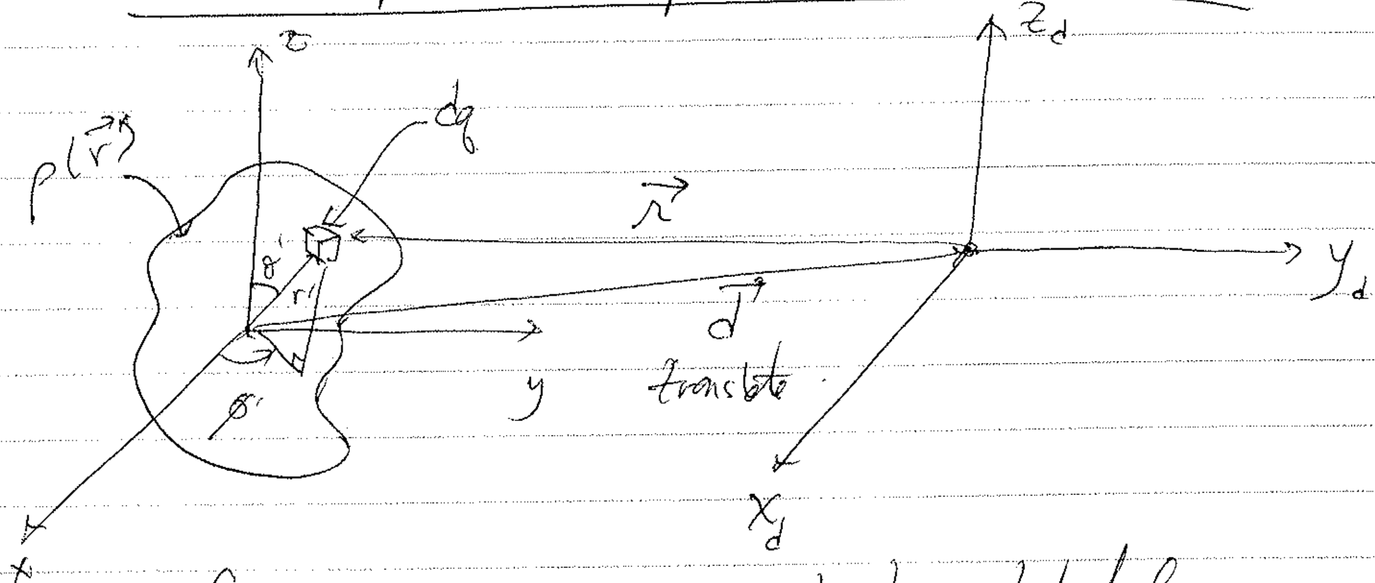
(viii) $Q_{31} = Q_{13} = 0$

(ix) $Q_{32} = Q_{23} = 0$

$z=0$ for all charges

So, we have $Q_{12} = Q_{21} = -3qa^2$ and $Q_{ij} = 0$ if otherwise

⑨ Show that Q_{ij} is independent of origin if the monopole and dipole moments vanish



find moment w/respect to translated frame

$$Q_{ij} = \int \left[3x_i' x_j' - r'^2 \delta_{ij} \right] \rho(\vec{r}')\, d^3x'$$

$$\Rightarrow Q_{ij}^d = \int \left[3(x_i' - d_i)(x_j' - d_j) - |(\vec{r}' - \vec{d})^2| \delta_{ij} \right] \underbrace{\rho(\vec{r}' - \vec{d})}_{\text{scalar and unaffected by translation}}\, \underbrace{d^3(x_i' - d_i)}$$

where $\vec{d} = (d_1, d_2, d_3)$ for above

$d(x_1' - d_1) = dx_1'$, ...

$\Rightarrow d^3x'$ is unaffected by translation

look at $\left[3(x_i' - d_i)(x_j' - d_j) - \delta_{ij} |\vec{r}' - \vec{d}|^2 \right]$

$$\Rightarrow \left[3\left\{ x_i' x_j' - x_i' d_j - d_i x_j' + d_i d_j \right\} - \delta_{ij} \left\{ \underbrace{(x_1' - d_1)^2 + (x_2' - d_2)^2 + (x_3' - d_3)^2}_{(x_1'^2 + x_2'^2 + x_3'^2) + (d_1^2 + d_2^2 + d_3^2) - 2x_1' d_1 - 2x_2' d_2 - 2x_3' d_3} \right\} \right]$$

grasp the terms in an illustrative fashion

$$\left[\underbrace{\cancel{3}\left[3x_i'x_j'-\delta_{ij}r'^2\right]}_{\Rightarrow Q_{ij}\text{ for original frame}}\right]+\left[\underbrace{3d_id_j+d^2\delta_{ij}}_{\text{independent of }\vec{r}'}\right]+\left[\underbrace{-3\left(d_jx_i'+d_ix_j'\right)}\underbrace{-2\vec{d}\cdot\vec{x}'\delta_{ij}}\right]$$

$$\Rightarrow (3d_id_j+\delta_{ij}d^2)\underbrace{\int\rho(\vec{r}')d^3x'}_{Q\,=\,\text{monopole moment}}$$

$$-2\vec{d}\cdot\int\rho(\vec{r}')\vec{x}'\delta_{ij}d^3x'$$

$$-2\delta_{ij}\vec{d}\cdot\underbrace{\int\rho(\vec{r}')\vec{x}'d^3x'}_{\vec{p}\,=\,\text{dipole moment}}$$

$$\underbrace{-3d_j\int\rho(\vec{r}')x_i'd^3x'-3d_i\int\rho(\vec{r}')x_j'd^3x'}$$

$-3d_jP_i-3d_iP_j$ where ~~where~~ $\vec{p}=(P_1,P_2,P_3)$

So, if $Q=0$, $\vec{P}=0 \Longrightarrow Q_{ij}^{d}=Q_{ij}$

as Q term dropout, $\vec{p}$ term drops out and the P_i, P_j terms dropout.

(d) $V_{octopole} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^4} \int \underbrace{r'^3 P_3(\cos\theta')}\, \rho(\vec{r}')\, d^3x'$

$= r'^3 \left[\frac{1}{2} \left(5\cos^3\theta' - 3\cos\theta' \right) \right]$

$= r'^3 \left[\frac{1}{2} \left(5(\hat{r}\cdot\hat{r}')^3 - 3(\hat{r}\cdot\hat{r}') \right) \right]$

$V_{octopole} = \frac{1}{4\pi\epsilon_0^3} \frac{1}{2r^4} \int r'^3 \left[5(\hat{r}\cdot\hat{r}')^2 - 3 \right] (\hat{r}\cdot\hat{r}')\, \rho(\vec{r}')\, d^3x'$