Physics 412: Introduction to Electrodynamics
Homework 8
Due: Friday, 2012 November 30

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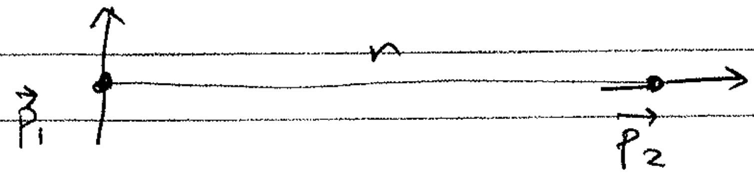
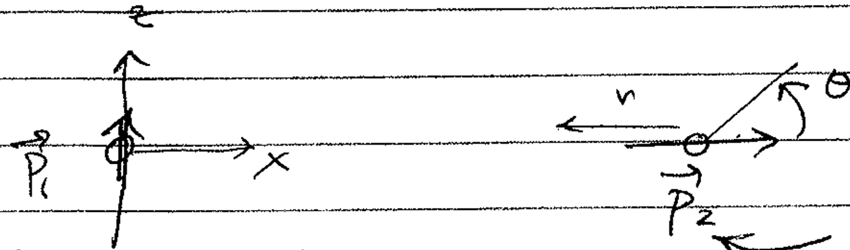
Prob 4.5



a) Find the torque on $\vec{p}_1$ due to $\vec{p}_2$



$\vec{E}_2 = \frac{p_2}{4\pi\epsilon_0 r^3}\left(-2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$

$= \frac{p_2}{4\pi\epsilon_0 r^3}\left(2\hat{r} - 0\hat{\theta}\right)$

$\vec{p}_1 = p_1(0,0,1)$

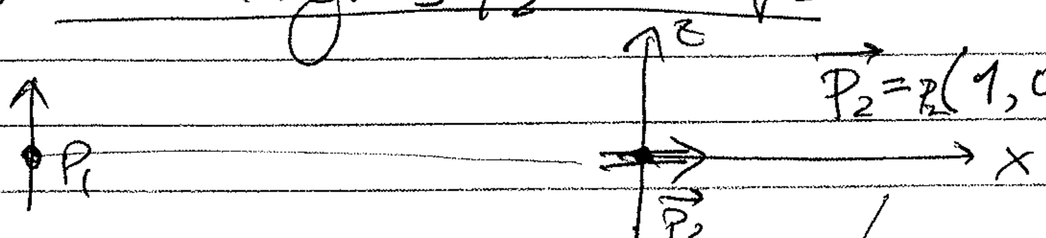
in Cartesian system, $\vec{E}_2' = \frac{p_2}{4\pi\epsilon_0 r^3}(2,0,0)$ — $\hat{x} : \hat{y} : \hat{z}$

$$\Rightarrow \boxed{\vec{N} = \vec{p}_1 \times \vec{E}_2' = (0,2,0)\frac{p_1 p_2}{4\pi\epsilon_0 r^3}}$$

$+\hat{y}$ is into the paper

$\Rightarrow \vec{p}_1$ rotates in the clockwise sense

b) Find an torque of $\vec{p}_2$ due to $\vec{p}_2$



$\vec{p}_2 = p_2(1,0,0)$

$\vec{E}_1 = \frac{p_1}{4\pi\epsilon_0 r^3}\left(0\hat{r} + \hat{\theta}\right)$

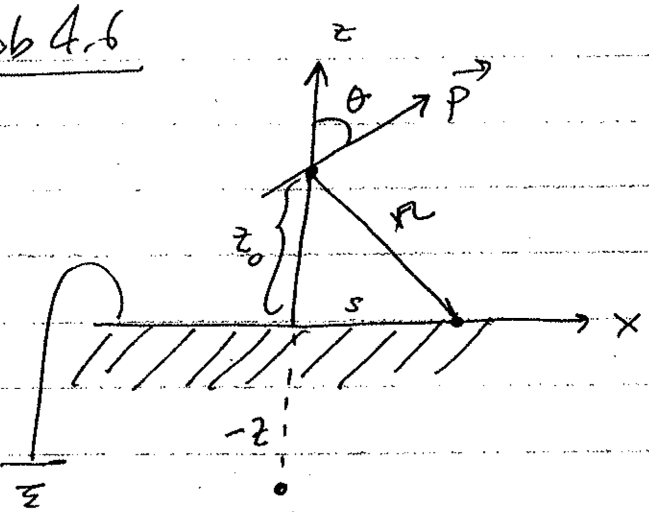
in cartesian coordinates

$\vec{E}_1' = \frac{p_1}{4\pi\epsilon_0 r^3}(0,0,-1)$

$\vec{N} = \vec{p}_2 \times \vec{E}_1'$

$$\boxed{\vec{N} = \frac{p_1 p_2}{4\pi\epsilon_0 r^3}(0,+1,0)}$$

Prob 4.6



Find the torque on $\vec{p}$

Sol^n

I) Q: where should we place the image dipole?

A: We should place it at $-z$, but at what angle?

let $\vec{p} = p(\sin\theta\,\hat{x} + \cos\theta\,\hat{z})$

(i) $V = \dfrac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} = \dfrac{p}{4\pi\epsilon_0 r^2}(\sin\theta\,\hat{x} + \cos\theta\,\hat{z})\cdot\dfrac{(x-0,\, y-0,\, z-z_0)}{r}$

$= \dfrac{p}{4\pi\epsilon_0 r^3}\left(x\sin\theta + (z-z_0)\cos\theta\right)$

(ii) at plane, $V + V_i = 0$ because conductor is grounded.

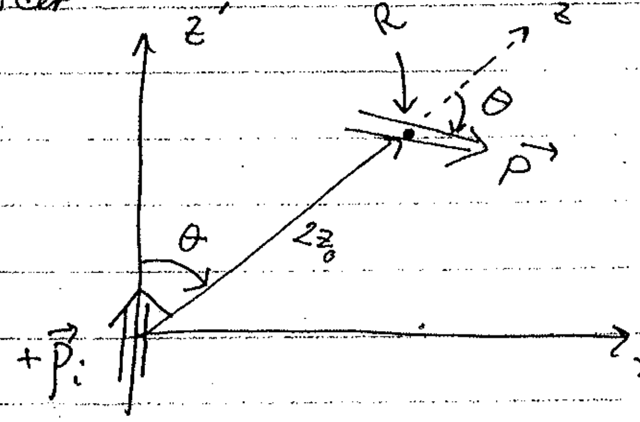
$\Longrightarrow V_i = -\dfrac{p}{4\pi\epsilon_0 r^2}\left(x\sin\theta + (\cancel{z}^{\,0} - z_0)\cos\theta\right)$

$= \dfrac{p}{4\pi\epsilon_0 r^2}\left(x\sin\theta_i + (\cancel{z}^{\,0} + z_0)\cos\theta_i\right)$

$\Longrightarrow x(\sin\theta_i + \sin\theta) = z_0(-\cos\theta_i + \cos\theta)$

to be true in general $\Longrightarrow \theta_i = -\theta$

Find the torque of the image $\vec{p}_i$ on the real $\vec{p}$.
Consider



rotate the axes by $-\theta$ degrees so that the image $\vec{p}_i$ points along $\hat{z}'$. Translate the axes by z so that $\vec{p}_i$ sits at the origin.

so, $\vec{E}_i = \frac{|\vec{p}|}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$. Find the torque $\vec{N}$ around R.

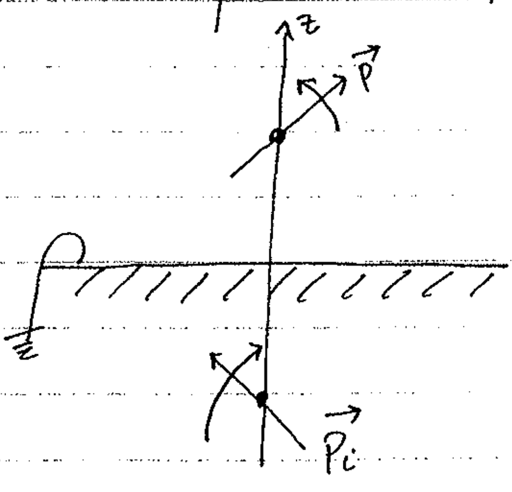
$$\Rightarrow \vec{N} = \vec{p} \times \vec{E}_i$$

$$= |\vec{p}|\left(\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \times \frac{|\vec{p}|}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$= \frac{|\vec{p}|^2}{4\pi\epsilon_0 r^3}\left[\sin\theta\cos\theta - 2\sin\theta\cos\theta\right]\hat{y}'$$ ← about y'-axis

$$\vec{N} = -\frac{|\vec{p}|^2}{8\pi\epsilon_0 r^3}\sin 2\theta\,\hat{y}'$$ ⇒ tries to rotate $\vec{p}$ to z-axis

(II) the dipole wants to arrange itself so that it finds itself in a stable equilibrium.



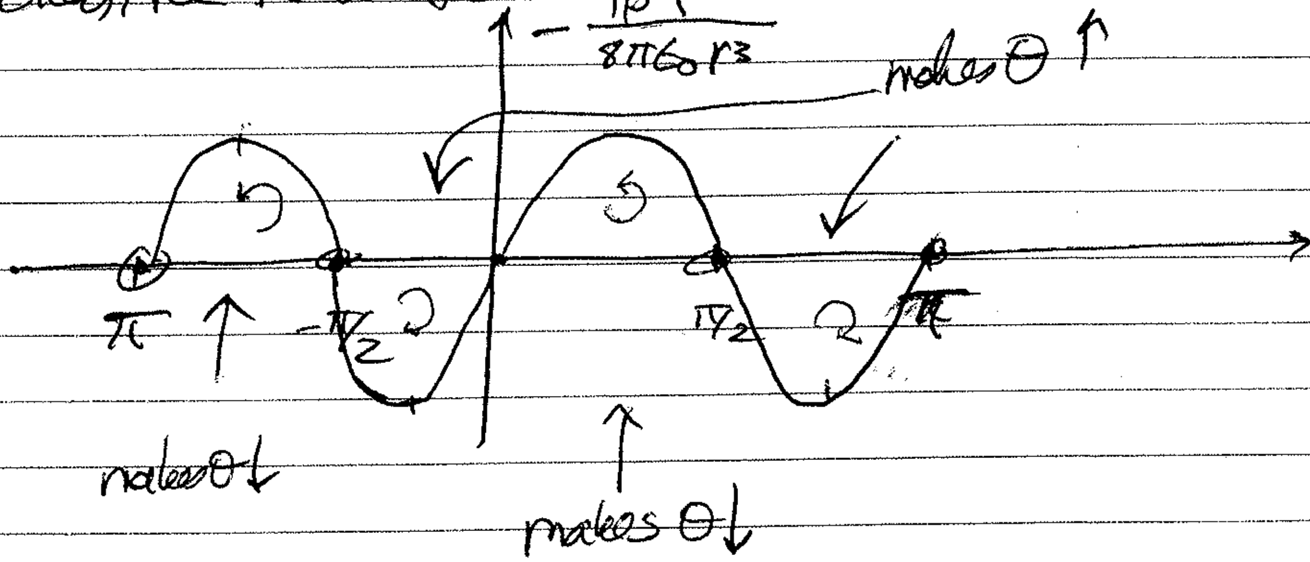
Let's check out the stability.

a) Torque, $\vec{N} = -\frac{|\vec{p}|^2}{4\pi\epsilon_0 r^3}\left[\sin 2\theta\, \hat{y}'\right]$

$\hat{y}'$ is into the paper (see previous page) $\Longrightarrow$

$\Longrightarrow$ $\vec{N} < 0$ means a rotation toward the $\hat{z}'$ axis or counterclockwise (decreases θ).

okay, the $\vec{N}$ is then



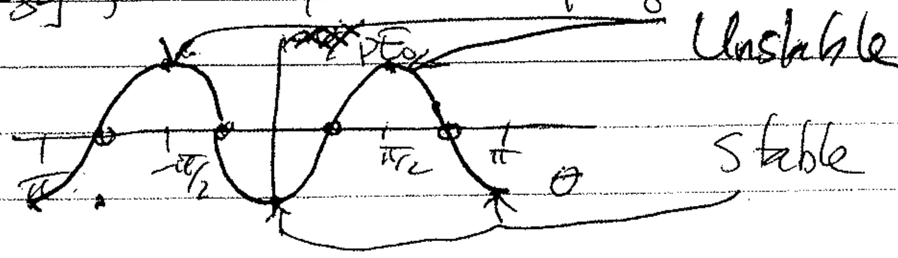
b) So if $0 < \theta < \pi/2$, $\vec{N}$ causes $\theta\downarrow$

if $\pi/2 < \theta < \pi$, $\vec{N}$ causes $\theta\uparrow$

c) $\Longrightarrow$ θ is driven to $0, \pi$, depending upon if its starts out $\lessgtr \pi/2$.

why 2θ?

b) Energy, $U = -\vec{p}\cdot\vec{E} = -pE_0\cos 2\theta$



Prob. 4.7

Show that the dipole interaction energy is

$$U = -\vec{p}\cdot\vec{E}$$

Suppose we has a small localized charge given by $q = \int \rho(\vec{r})\, d^3x$. The interaction energy w/a field $V(\vec{r})$ is then

$$U = \int \rho(\vec{r}')\, d^3x'\, V(\vec{r}')$$

If the charge q is small, expand the potential about some center, "O", using a Taylor series

$$V(\vec{r}') = V(0) + \vec{x}'\cdot\vec{\nabla}V\Big|_0 + \cdots$$

evaluated at the "center", O.

only keep up to 1st order for dipole case

$$\Rightarrow U = \int \rho(\vec{r}')\, V(0)\, d^3x' + \int \rho(\vec{r}')\, \vec{x}'\cdot\vec{\nabla}V\Big|_0 d^3x' + \cdots$$

$$= \left[\int \rho(\vec{r}')\, d^3x'\right] V(0) + \left[\int \rho(\vec{r}')\, \vec{x}'\, d^3x'\right]\cdot\vec{\nabla}V\Big|_0 + \cdots$$

$$= Q\,V(0) + \vec{p}\cdot\vec{\nabla}V\Big|_0 + \cdots$$

$$= \underbrace{Q\,V(0)}_{=0 \text{ for a pure dipole}} - \vec{p}\cdot\vec{E}\Big|_0 + \underbrace{\cdots}_{\text{other multipoles } = 0 \text{ if pure dipole}}$$

$$\boxed{U = -\vec{p}\cdot\vec{E}}$$

Prob 4.10

A sphere carries polarization $\vec{P} = k\vec{r}$, where k is a constant.

a) Calculate σ_b and ρ_b.
b) Find the field inside and outside the sphere.

Sol^n

a) $\sigma_b = \vec{P}\cdot\hat{r} = (k\vec{r})\cdot\hat{r} = kR \leftarrow$ at surface $r = R$

b) $\rho_p = -\vec{\nabla}\cdot\vec{P} = -\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 k r\right) = -3k$

c) Find the field for $r < R$ and $r > R$. By symmetry, Gauss's law would be appropriate.

(i) $$\oint \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int \rho_p \, d\tau$$

$$E_r \, 4\pi r^2 = \frac{1}{\epsilon_0}\int_0^r (-3k)\, 4\pi r^2 dr, \quad r < R$$

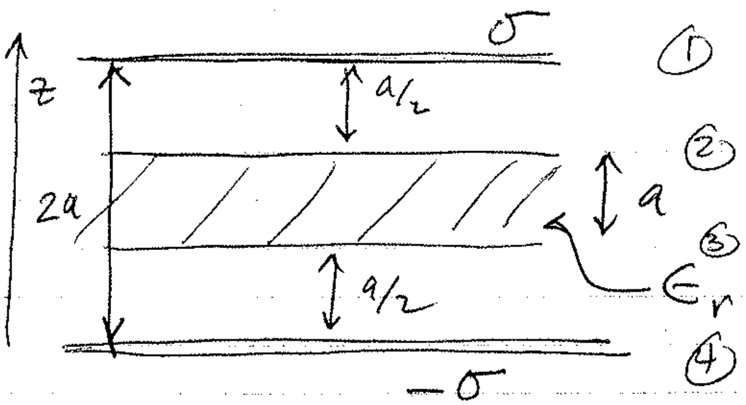
$$= -\frac{4\pi k}{\epsilon_0} r^3$$

$$\longrightarrow \vec{E}_r = -\frac{k}{\epsilon_0} r\hat{r} \qquad r < R$$

(ii) $$E_r \, 4\pi r^2 = \frac{1}{\epsilon_0}\left[\int_0^R \underbrace{(-3k)4\pi r^2 dr}_{-4\pi k R^3} + \oint kR\,4\pi R^2\right]$$

$$= 0 \qquad r > R$$

Problem 4.19



a) How is the capacitance changed?

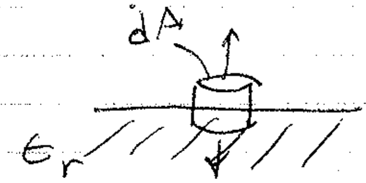
(i) $C_{vacuum} = \left(\dfrac{\epsilon_0 A}{2a}\right)$

(ii) find "new" capacitance

at ①, $\vec{E} = -\dfrac{\sigma}{\epsilon_0}\hat{z} = -\dfrac{V_0}{2a}\hat{z} = \dfrac{\vec{D}}{\epsilon_0}$ ← vacuum

at ②

(i)



$\oint \vec{D}\cdot d\vec{S} = 0$, no free charge

$D^+A - D^-A = 0 \rightarrow D^+ = D^-$

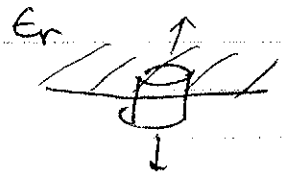
$\Rightarrow \epsilon_0 E^+ = \epsilon_0\epsilon_r E^-$

and $E_0^- = \dfrac{1}{\epsilon_r}E^-$

$= -\dfrac{\sigma \hat{z}}{\epsilon_0\epsilon_r}$

$= -\dfrac{\sigma}{\epsilon}\hat{z}$

at ③

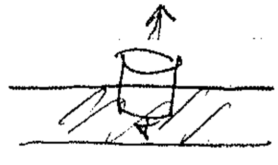


$D^+A - D_V^-A = 0 \rightarrow D^+ = D_V^-$

$-\sigma = D_V^-$

$\Rightarrow E_V = -\dfrac{\sigma}{\epsilon_0}$

at ④



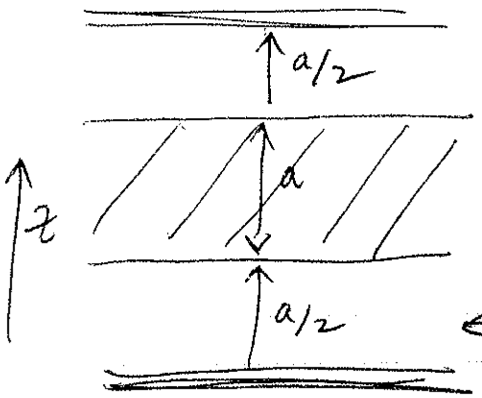
$\oint \vec{E}\cdot d\vec{S} = 0$

$\cancel{D}\ \overbrace{E^+A - 0 = -\sigma/\epsilon_0}$

$E_V = -\sigma/\epsilon_0$

$\vec{D}$ is the same in all regions,
but $\vec{E}$ changes according to ϵ

okay, gather up results



$$E_z = -\frac{\sigma}{\epsilon_0}, \quad D_z = -\sigma, \quad P_z = 0$$

$$E_z = -\frac{\sigma}{\epsilon}, \quad D_z = -\sigma, \quad P_z = \sigma\left(\frac{\epsilon_0}{\epsilon} - 1\right)$$

$$E_z = -\frac{\sigma}{\epsilon_0}, \quad D_z = -\sigma, \quad P_z = 0$$

find ΔV = potential difference.

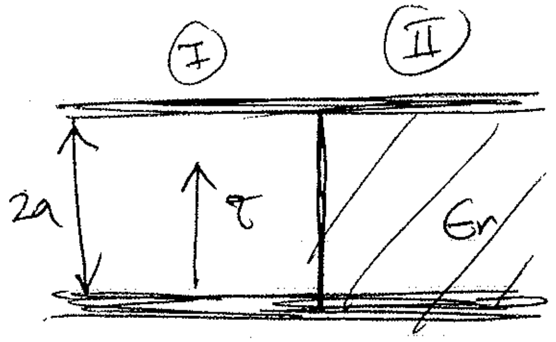
$$\Delta V = V_0 = -\int \vec{E} \cdot d\vec{z} = -\left[\int_0^{\frac{a}{2}} -\frac{\sigma}{\epsilon_0}\,dz + \int_{\frac{a}{2}}^{\frac{3a}{2}} -\frac{\sigma}{\epsilon}\,dz + \int_{\frac{3a}{2}}^{2a} -\frac{\sigma}{\epsilon_0}\,dz\right]$$

$$= \frac{\sigma}{\epsilon_0}\left[\frac{a}{2} + \left(\frac{\epsilon_0}{\epsilon}\right)\left(\frac{3a}{2} - \frac{a}{2}\right) + \left(2a - \frac{3a}{2}\right)\right]$$

$$V_0 = \frac{\sigma}{\epsilon_0}\left[a + a\left(\frac{\epsilon_0}{\epsilon}\right)\right] = \frac{\sigma a}{\epsilon_0}\left(\frac{\epsilon + \epsilon_0}{\epsilon}\right)$$

$$\Rightarrow \boxed{C_\epsilon = \frac{Q}{V_0} = \frac{\cancel{\sigma} A}{\frac{\cancel{\sigma} a}{\epsilon_0}\left(\frac{\epsilon+\epsilon_0}{\epsilon}\right)} = \frac{\epsilon \epsilon_0 A}{a(\epsilon + \epsilon_0)}}$$

$$\Rightarrow \frac{C_\epsilon}{C_{vac}} = \frac{\cancel{\epsilon_0} \epsilon \cancel{A}}{\cancel{a}(\epsilon + \epsilon_0)}\left(\frac{2\cancel{a}}{\cancel{\epsilon_0}\cancel{A}}\right) = \left(\frac{2\epsilon}{\epsilon + \epsilon_0}\right)$$



What is the capacitance?

a) Because the top plate is an equipotential and the bottom plate is an equipotential, and $\rho_f = \rho_p = 0$;

$$\vec{E} = -\frac{V_0}{2a}\hat{z}$$

where V_0 is the potential difference, in both regions

b) But now, what is Q?

(i) $E_I = -\frac{\sigma_I^f + \sigma_I^p}{\epsilon_0}$; $E_{II} = -\frac{\sigma_{II}^f + \sigma_{II}^p}{\epsilon_0} = E_I$

(ii) $D_I = -\frac{\sigma_I^f}{\epsilon_0}$; $D_{II} = -\sigma_{II}^f = \epsilon_r \epsilon_0 E_{II}$

$$\Rightarrow E_{II} = -\frac{\sigma_{II}^f}{\epsilon_r \epsilon_0}$$

combine (i) and (ii)

$$E_{II} = -\frac{\sigma_{II}^f + \sigma_{II}^p}{\epsilon_0} = -\frac{\sigma_{II}^f}{\epsilon_0 \epsilon_r} \Rightarrow \sigma_{II}^p = \sigma_{II}^f\left(\frac{1}{\epsilon_r} - 1\right)$$

$$\to E_I = -\frac{\sigma_I^f}{\epsilon_0} = -\frac{\sigma_{II}^f}{\epsilon_0} - \frac{\sigma_{II}^f}{\epsilon_0}\left(\frac{1}{\epsilon_r} - 1\right) = -\frac{\sigma_{II}^f}{\epsilon_r \epsilon_0}$$

and so, $+\sigma_I^f = \frac{1}{\epsilon_r}\sigma_{II}^f$

I	II
$\vec{E}_z = -\frac{V_0}{2a}\hat{z}$, $\vec{D} = \epsilon_0 \vec{E}_z$, $\vec{P}_z = 0$	$\vec{E}_z = -\frac{V_0}{2a}\hat{z}$, $\vec{D} = \epsilon \vec{E}_z$, $\vec{P}_z = -\frac{V_0}{2a}\hat{z}(\epsilon - \epsilon_0)$

Capacitance, $C = \dfrac{Q}{V_0}$, is then,

$$C_\epsilon = \frac{\sigma_I^f\left(\frac{A}{2}\right) + \sigma_{II}^f\left(\frac{A}{2}\right)}{V_0}$$

$$= \left(\frac{\sigma_I^f + \epsilon_r \sigma_I^f}{V_0}\right)\frac{A}{2}$$

note: $E = -\dfrac{V_0}{2a} = -\dfrac{\sigma_I^f}{\epsilon_0} \Longrightarrow \sigma_I^f = \dfrac{\epsilon_0 V_0}{2a}$

$$C_\epsilon = \frac{1+\epsilon_r}{\cancel{V_0}}\,\frac{A}{2}\,\frac{\epsilon_0 \cancel{V_0}}{2a}$$

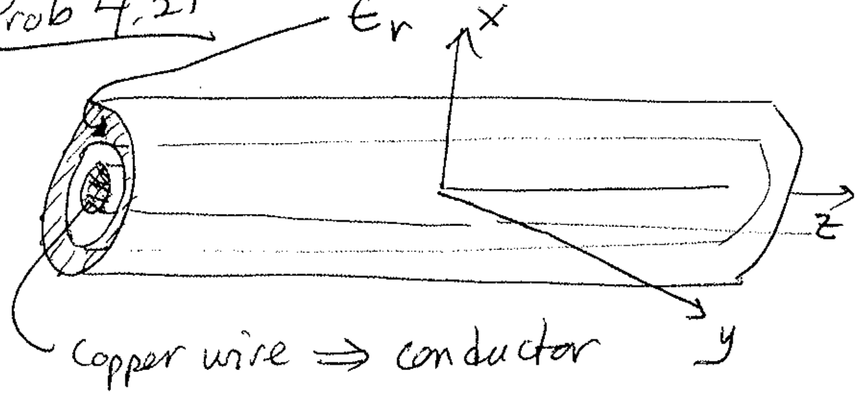
$$= \boxed{\frac{\cancel{\epsilon_r}\epsilon_0(1+\epsilon_r)A}{4a}}$$

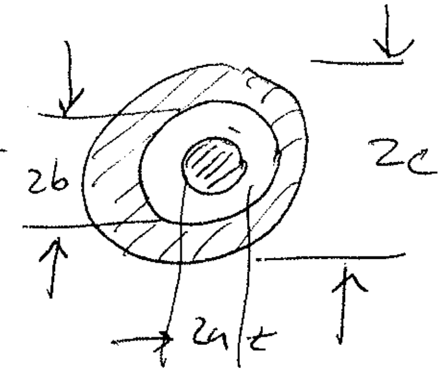
$$\frac{C_\epsilon}{C_{vac}} = \frac{\epsilon_0\cancel{A}(1+\epsilon_r)}{4\cancel{a}}\,\frac{2\cancel{a}}{\epsilon_0\cancel{A}} = \frac{\epsilon_0(1+\epsilon_r)}{\epsilon_0 2}$$

$$\boxed{\frac{C_\epsilon}{C_{vac}} = \left(\frac{1+\epsilon_r}{2}\right) = \frac{\epsilon+\epsilon_0}{2\epsilon_0}}$$

$\boxed{\text{For } \vec{E}, \vec{D}, \vec{P}, \text{ see the circled boxes}}$

Prob 4.21





Find the capacitance of this coaxial cable.

a) the copper wire has uniform potential V_0 (because it is a conductor). In cylindrical conductors the Laplace Equation is

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2}{\partial \phi^2}V + \frac{\partial^2}{\partial z^2}V$$

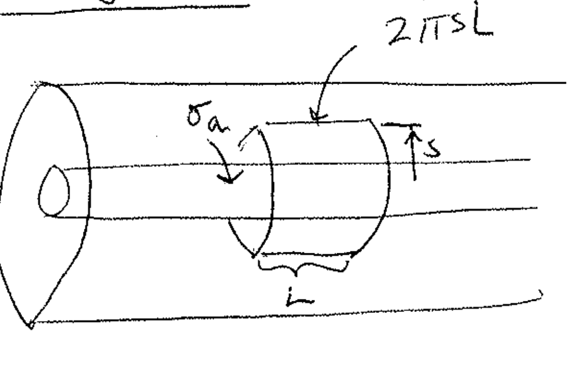
which reduces to

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$ for the infinite coaxial cable

$$\Rightarrow \boxed{V = C_0 + C_1 \ln s}$$

Set $V = V_0$ at $s = a \Rightarrow$

b) Solve for $\vec{D}$



$$\Rightarrow D_s 2\pi s L = \sigma 2\pi a L$$

$$\boxed{D_s = \sigma\left(\frac{a}{s}\right)}$$

i) $a < s < b$

$$D_s = \sigma\left(\frac{a}{s}\right) \; \& \; E_s = \frac{\sigma a}{\epsilon_0 s}$$

(ii) $b < s < C$

$$D_s = \sigma\left(\frac{a}{s}\right) \quad \& \quad E_s = \frac{\sigma a}{\epsilon_0 \epsilon_r s}$$

c) <u>Solve for V</u>

$$\int dV = -\vec{E}\cdot d\vec{s}$$

$$V(c) - V(a) = -\int_a^b \frac{\sigma a}{\epsilon_0 s}\, ds - \int_b^c \frac{\sigma a}{\epsilon_0 \epsilon_r s}\, ds$$

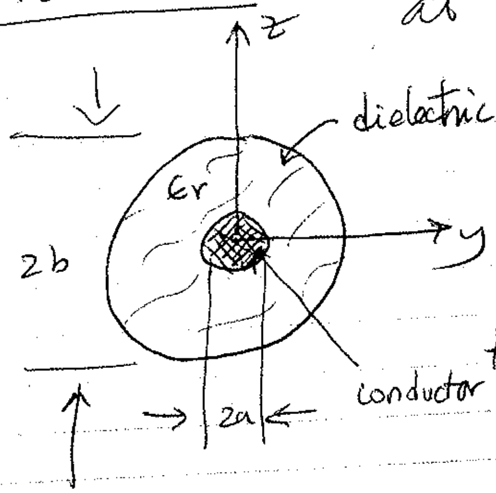
$$= -\frac{\sigma a}{\epsilon_0} \ln\left(\frac{b}{a}\right) - \frac{\sigma a}{\epsilon_0 \epsilon_r} \ln\left(\frac{c}{b}\right)$$

$$\underbrace{V(c) - V(a)}_{\Delta V} = -\frac{\sigma a}{\epsilon_0}\left[\ln\left(\frac{b}{a}\right) + \frac{1}{\epsilon_r} \ln\left(\frac{c}{b}\right)\right]$$

d) $$C = \frac{Q}{\Delta V} = \left|\frac{\sigma 2\pi a L}{\Delta V}\right| = \frac{\epsilon_0}{\cancel{a}} \frac{\cancel{\Delta V}\, 2\pi \cancel{a} L}{\ln\left(\frac{b}{a}\right) + \frac{1}{\epsilon_r} \ln \frac{c}{b}} \frac{1}{\cancel{\Delta V}}$$

$$\Rightarrow \boxed{\frac{C}{L} = \frac{2\pi \epsilon_0}{\ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b}}}$$

Prob 4.24



at $r=\infty$, $\vec{E}=E_0 z \rightarrow V=-E_0 r\mu + V_\infty$

there is no free charge anywhere, except for charge which can be induced on the surface of the conductor.

Find the Electric field in the insulator.

Sol$^{\underline{n}}$

ⓐ BCs ⓐ at $r=a$, $V=V_0 \equiv$ constant on conductor

ⓑ at $r=b$, V is continuous and $\Delta D_r = 0$

ⓒ at $r=\infty$, $V=-E_0 r\mu + V_\infty$

ⓑ find the solution in $a<r<b$ & $b<r<\infty$ to avoid charged shells at $r=a$ and $r=b$.

Use the sol$^{\underline{n}}$ to the Laplace equation,

$$V(r,\mu) = \sum_{\ell=0} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\mu)$$

ⓒ Outer Solution, $r>b$

$$V^> = \sum_{\ell=0} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\mu) \Rightarrow A_\ell = 0 \text{ except for } \ell=0,1$$

at $\ell=0,1$; $A_0 = V_\infty$, $A_1 = -E_0$

$$\Rightarrow \boxed{V^>(r,\mu) = V_\infty - E_0 r\mu + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\mu)}$$

d) Inner Solution, $b > r > a$

$$V^{<} = \sum_{\ell=0}^{\infty}\left(A'_\ell r^\ell + \frac{B'_\ell}{r^{\ell+1}}\right)P_\ell(u)$$

(i) at $r=b$, $V^{<} = V^{>}$

$$\Rightarrow V_\infty - E_0 b u + \sum_{\ell=0}^{\infty}\frac{B_\ell}{b^{\ell+1}}P_\ell(u) = \sum_{\ell=0}^{\infty}\left(A'_\ell b^\ell + \frac{B'_\ell}{b^{\ell+1}}\right)P_\ell(u)$$

match terms w/ same $P_\ell(u)$

(1) $\boxed{V_\infty = A'_0 + B'_0/b}$

(2) $\boxed{-E_0 b + \frac{B_1}{b^2} = A'_1 b + \frac{B'_1}{b^2}}$

(3) $\frac{B_2}{b^3} = A'_2 b^2 + \frac{B'_2}{b^3}$

(4) $\vdots$ $\boxed{\frac{B_\ell}{b^{\ell+1}} = A'_\ell b^\ell + \frac{B'_\ell}{b^{\ell+1}}}$; in general for $\ell \neq 0,1$

(ii) at $r=b$, $\Delta D_r = 0 = -\epsilon_0\frac{\partial V^{>}}{\partial r} - \left(-\epsilon\frac{\partial V^{<}}{\partial r}\right) = 0$

$$0 = -\epsilon_0\left[-E_0 u - \sum_{\ell=0}^{\infty}(\ell+1)\frac{B_\ell}{b^{\ell+2}}P_\ell(u)\right] + \epsilon\left[\sum_{\ell=0}^{\infty}\left(\ell A'_\ell b^{\ell-1} - (\ell+1)\frac{B'_\ell}{b^{\ell+2}}\right)P_\ell(u)\right]$$

match terms w/ same $P_\ell(u)$

(1) $\epsilon_0\frac{B_0}{b^2} - \epsilon\frac{B'_0}{b^2} = 0 \Longrightarrow \boxed{B'_0 = (\epsilon_0/\epsilon)B_0}$

(2) $\boxed{\epsilon_0 E_0 + \epsilon_0 2\frac{B_1}{b^3} + \epsilon A'_1 - 2\epsilon\frac{B'_1}{b^3} = 0}$

in general case,

$$③\ \boxed{\epsilon_0 (l+1)\frac{B_l}{b^{l+2}} + \epsilon\left(l A_l' b^{l-1} - (l+1)\frac{B_l'}{b^{l+2}}\right) = 0}$$ for all $l > 0$

(iv) at $r=a$, $V^< = V_0 \equiv$ constant

$$V_0 = \sum_{l=0}^{\infty}\left(A_l' a^l + \frac{B_l'}{a^{l+1}}\right) P_l(\mu)$$

$$① \quad V_0 = (A_0' + B_0'/a) \Longrightarrow \boxed{A_0' = V_0 - \frac{B_0'}{a}}$$

$$② \quad A_l' a^l + B_l'/a^{l+1} = 0 \Longrightarrow \boxed{A_l' = -\frac{B_l'}{a^{2l+1}}}$$

$$\Longrightarrow V^<(r,\mu) = \underbrace{\left[\left(V_0 - \frac{B_0'}{a}\right) + \frac{B_0'}{ar}\right]}_{l=0} + \sum_{l=1}^{\infty}\left(-\frac{r^l}{a^{2l+1}} + \frac{1}{r^{l+1}}\right) B_l' P_l$$

iv) wait, we set $V_\infty \neq 0$ and $V_0 \neq 0$ (at conductor) but we see that $V_0 = V_\infty$ from the above conditions?

Let's set $V_0 = V_\infty$ for simplicity

$$\Rightarrow A_0' = -B_0'/b$$

Okay, now let's gather conditions

$\underline{\ell = 0}$ $A_0' + B_0'/b = 0$ & $A_0 = V_\infty = 0$ (A)

$\underline{\ell = 1}$ $-bE_0 + \frac{1}{b^2} B_1 = b A_1' + \frac{1}{b^2} B_1'$ (B)

$$\epsilon_0 E_0 + \frac{2\epsilon_0}{b^3} B_1 = -\epsilon A_1' + 2\epsilon \frac{B_1'}{b^3} \quad (C)$$

$\underline{\ell \neq 0, 1}$

$$\frac{1}{b^{\ell+1}} B_\ell = A_\ell' b^\ell + \frac{B_\ell'}{b^{\ell+1}} \quad (D)$$

$$\epsilon_0 \frac{(\ell+1)}{b^{\ell+2}} B_\ell = -\epsilon \left(\ell b^{\ell-1} A_\ell' - \frac{(\ell+1)}{b^{\ell+2}} B_\ell' \right) \quad (E)$$

and note that

$$A_\ell' = -\frac{B_\ell'}{a^{2\ell+1}} \quad (F)$$

The only way to satisfy (D) & (E) is to set $B_\ell' = 0$

$(\Rightarrow A_\ell' = B_\ell = 0)$

So then

(i) $A_0' + \frac{B_0'}{b} = 0 \Rightarrow A_0' = -\frac{B_0'}{b}$ & $\underbrace{A_0' = V_0 - \frac{B_0'}{a} \Rightarrow A_0' = -}_{\to A_0' = B_0' = 0}$

(ii) $$\begin{cases} -bE_0 + \frac{B_1}{b^2} = b\left(-\frac{B_1'}{a^3}\right) + \frac{B_1'}{b^2} \\ \epsilon_0 E_0 + \frac{2\epsilon_0}{b^3} B_1 = -\epsilon\left(-\frac{B_1'}{a^3}\right) + \frac{2\epsilon}{b^3} B_1' \end{cases}$$

$$\Rightarrow b^2\left[+bE_0 - \left(\frac{b}{a^3} - \frac{1}{b^2}\right) B_1'\right] = \frac{b^3}{2\epsilon_0}\left[-\epsilon_0 E_0 + \left(\frac{\epsilon}{a^3} + \frac{2\epsilon}{b^3}\right) B_1'\right.$$

$$B_1'\left(-\frac{b^3}{a^3}+1-\frac{\epsilon}{\epsilon_0}\left[\frac{b^3}{2a^3}+1\right]\right)=\left(-\frac{E_0 b^3}{2}-E_0 b^3\right)$$

$$B_1'=\frac{-\frac{3}{2}E_0 b^3}{\left(1-\epsilon_r+\frac{b^3}{a^3}\left[-1-\frac{\epsilon_r}{2}\right]\right)}=\frac{-\frac{3}{2}E_0 b^3}{(1-\epsilon_r)-\left(1+\frac{\epsilon_r}{2}\right)\frac{b^3}{a^3}}$$

$$\Rightarrow -A_1'=a^{-3}\left[\frac{-\frac{3}{2}E_0 b^3}{(1-\epsilon_r)-\left(1+\frac{\epsilon_r}{2}\right)\frac{b^3}{a^3}}\right]$$

$$A_1'=\frac{\frac{3}{2}E_0 (b/a)^3}{(1-\epsilon_r)-(1+\epsilon_r/2)(b/a)^3}$$

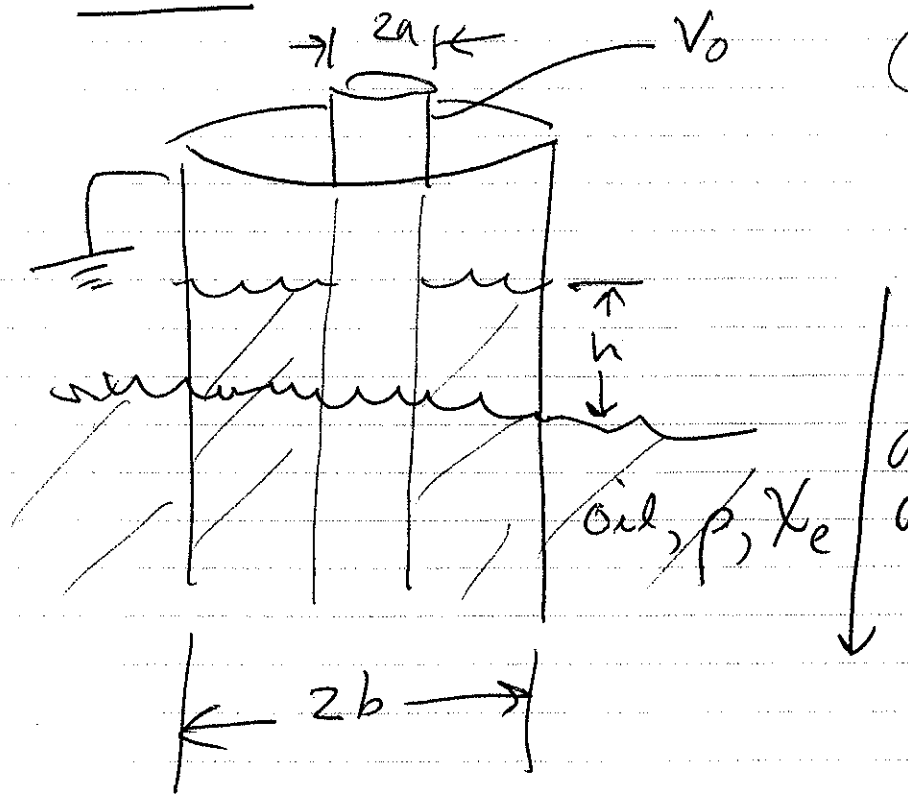
ad so,

$$\boxed{V^{<}(r,\mu)=\left[\frac{\frac{3}{2}E_0 (b/a)^3}{(1-\epsilon_r)-\left(1+\frac{\epsilon_r}{2}\right)\left(\frac{b}{a}\right)^3}\right]\left(r-\frac{a^3}{r^2}\right)\mu}$$

ad

$$\vec{E}=-\vec{\nabla}V^{<}(r,\mu)$$

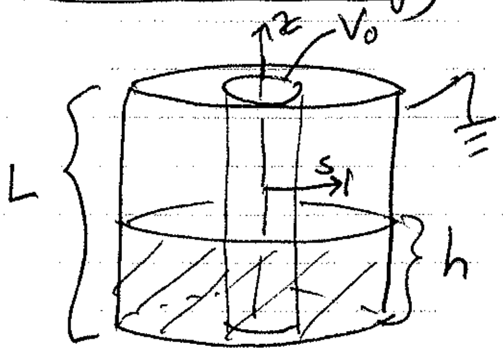
Prob 4.28



(a) find h, the height to which the oil rises.

We need to balance the tendency of the dielectric to be pulled into the field against the downward pull of gravity.

(b) Find the energy in the "capacitor."



(i) In vacuum region,

$$\vec{E}_v = -\vec{\nabla} V = -\vec{\nabla}\left[c_0 + c_1 \ln s\right]$$

where $V = c_0 + c_1 \ln s$ from Laplace eqn

at $s = a,\ V = V_0$; $s = b,\ V = 0$ $\Rightarrow$ $c_0 = -c_1 \ln b$, $c_0 = V_0 - c_1 \ln a$

and so
$$\begin{cases} c_1 = V_0 / \ln(a/b) \\ c_0 = -V_0 \dfrac{\ln b}{\ln(a/b)} \end{cases}$$

$$\Rightarrow \vec{E}_v = -\left[\frac{V_0}{\ln\left(\frac{a}{b}\right)} \frac{\hat{s}}{s}\right]$$

(ii) Note that because $\rho_p = 0$ in dielectric (because $\rho_f = 0$) that

$$\vec{E}_{Dielectric} = \vec{E}_V$$

(iii) In dielectric, however,

$$\vec{D} = \epsilon \vec{E}_{Dielectric} = \epsilon \vec{E}_V$$

(iv) Energy densities are then

Vacuum

$$\frac{\epsilon_0}{2} E_V^2 = \frac{\epsilon_0}{2}\left[\frac{V_0}{\ln\left(\frac{a}{b}\right)}\right]^2 \frac{1}{s^2}$$

Dielectric

$$\frac{1}{2}\vec{D}\cdot\vec{E}_V = \frac{\epsilon}{2}\left[\frac{V_0}{\ln\left(\frac{a}{b}\right)}\right]^2 \frac{1}{s^2}$$

(v) Total Energy is then

$$W_{Tot} = \int_h^L\iint_a^b \frac{\epsilon_0}{2}\left(\frac{V_0}{\ln(a/b)}\right)^2 \frac{1}{s^2}\, s\,ds\,d\phi\,dz + \int_0^h\iint_a^b \frac{\epsilon}{2}\left(\frac{V_0}{\ln\left(\frac{a}{b}\right)}\right)^2 \frac{1}{s^2}\, s\,ds\,d\phi\,dz$$

$$= \frac{1}{2}\left(\frac{V_0}{\ln\left(\frac{a}{b}\right)}\right)^2 2\pi \ln\left(\frac{b}{a}\right)\left[\epsilon_0(L-h) + \epsilon(h-0)\right]$$

(vi) Force is then (because $\Delta V = V_0$ is held fixed)

$$F_{\cancel{z}} = + \frac{dW}{d\cancel{h}} = \frac{+V_0^2\, 2\pi}{2\ln(b/a)}\left[L\epsilon_0 + h(\epsilon - \epsilon_0)\right]'$$

$$\boxed{F_z = \frac{\pi V_0^2}{\ln(b/a)}\left[\epsilon - \epsilon_0\right]}$$

(vii) <u>Gravity</u>:

$$F_z = -g\Big(\overbrace{\pi\left[b^2-a^2\right]h\rho}^{\text{mass of dielectric}}\Big)$$

$$\Longrightarrow \frac{\cancel{\pi} V_0^2}{\ln\left(\frac{b}{a}\right)}(\epsilon-\epsilon_0) - \cancel{\pi}(b^2-a^2)\rho g h = 0$$

$$\Longrightarrow \boxed{h = \frac{(\epsilon-\epsilon_0)V_0^2}{\rho g(b^2-a^2)\ln(b/a)}}$$