Test 1

Problem 1:

a) force ~~and~~ on charge $q = -e$ (electron). Assume the positive is spread uniformly so that

$$\rho_+(r) = \begin{cases} \text{constat} = \dfrac{3Ze}{4\pi R^3} & , r < R \\ = 0 & , r > R \end{cases}$$

Use Gauss's law of $\vec{E}$, also calculate $Q(r)$,

$$Q(r) = \begin{cases} \rho_+ \frac{4\pi}{3} r^3 & , r < R \\ Ze & , r > R \end{cases}$$

$$\Rightarrow \oint \vec{E} \cdot d\vec{S} = E_r 4\pi r^2 = Q(r)/\epsilon_0$$

$$\Rightarrow E_r = \begin{cases} \dfrac{\rho_+ \frac{4\pi}{3} r^3}{4\pi r^2 \epsilon_0} & , r < R \\ \dfrac{Ze}{4\pi\epsilon_0 r^2} & , r > R \end{cases}$$

$$= \begin{cases} \dfrac{Zer}{4\pi\epsilon_0 R^3} & , r < R \\ \dfrac{Ze}{4\pi\epsilon_0 r^2} & , r > R \end{cases}$$

Force = $\boxed{\vec{F}_r = -e\left(\dfrac{Zer}{4\pi\epsilon_0 R^3}\right)\hat{r} \quad , r < R}$

b) $$m_e \ddot{r} = -\frac{Ze^2}{4\pi\epsilon_0 R^3} r \Rightarrow \ddot{r} + \underbrace{\frac{Ze^2}{4\pi\epsilon_0 m_e R^3}} r = 0$$

$$\boxed{\omega^2 = \frac{Ze^2}{4\pi\epsilon_0 m_e R^3}}$$

Problem 2:

a) look at $\vec{\nabla} \times \vec{E}$

(i) $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(c[x-z]^2, 0, -c[x-z]^2\right)$

$= \left(0-0,\ 2c(x-z)(-1) - [-2c(x-z)],\ 0-0\right)$

$= \left(0, -2c(x-z) + 2c(x-z), 0\right)$

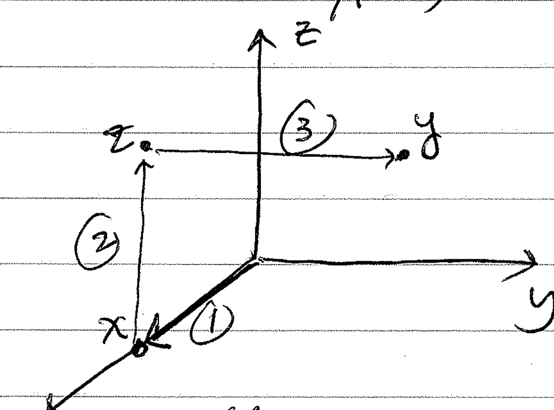
$= (0, 0, 0)$, conservative

(ii) $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(2xyz, xz^2, x^2y\right)$

$= \left(x^2 - 2xz,\ 2xy - 2xy,\ z^2 - 2xz\right)$

$\neq 0$, not conservative

b) Find $V(x,y,z)$



path

$$-\int_0^{(x,y,z)} dV = \underbrace{\int_0^x c x^2\, dx}_{(y=z=0)} + \underbrace{\int_0^z -c(x-z)^2\, dz}_{(x,y=0)} + 0$$

(1) (2) (3)

$$= \frac{c}{3}x^3 - \int_0^z c\left(x^2 - 2xz + z^2\right) dz$$

$$-V(x,y,z) + V_0 = \frac{c}{3}x^3 - cx^2z + cxz^2 - \frac{c}{3}z^3$$

$$\boxed{V(x,y,z) = cx^2z + \frac{c}{3}z^3 - \frac{c}{3}x^3 - cxz^2}$$

c) Find ρ responsible for conservative $\vec{E}$

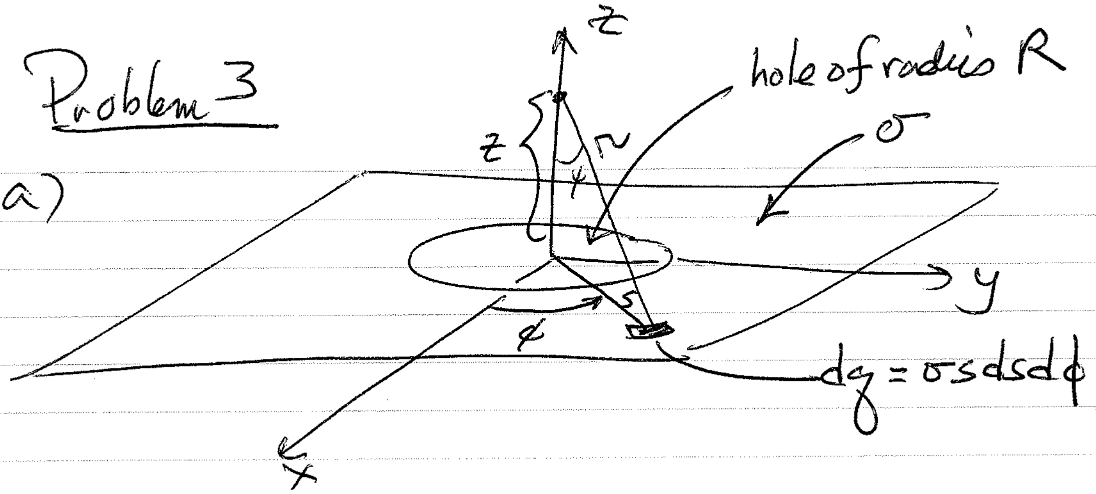
$$\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$$

$$= \epsilon_0 \left[2(x-z) - 2(x-z)(-1) \right]$$

$$= 4\epsilon_0 (x-z)$$

Problem 3

a)



By symmetry, only the z-component of $\vec{E}$ survives the integration.

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{\sigma s\,ds\,d\phi}{r^2}(\hat{r}\cdot\hat{z}) \qquad \hat{r}\cdot\hat{z} = \cos\psi = \frac{z}{r}$$

$$= \frac{\sigma z}{4\pi\epsilon_0}\left[\frac{s\,ds\,d\phi}{r^3}\right]$$

$$\Rightarrow E_z = \frac{\sigma z}{4\pi\epsilon_0}\int_0^{2\pi} d\phi \int_R^{\infty} \frac{s\,ds}{(z^2+s^2)^{3/2}}$$

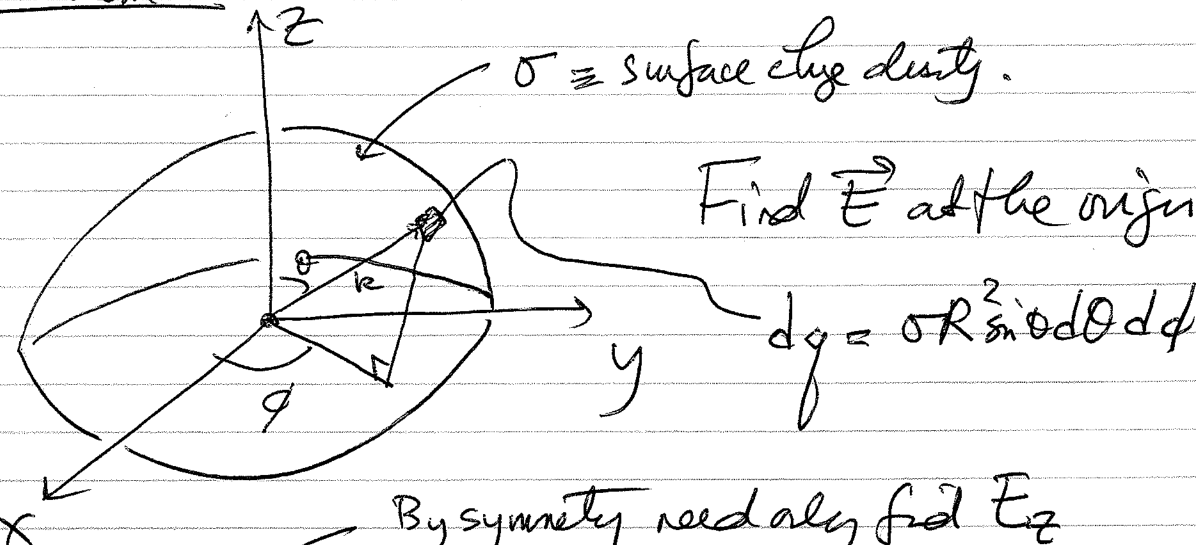
Let $X = z^2+s^2 \Longrightarrow dX = 2s\,ds$

$$= \frac{\sigma z}{2\epsilon_0}\int_{X_R}^{\infty}\frac{1}{2}\frac{dX}{X^{3/2}}$$

$$= -\frac{\sigma z}{4\epsilon_0}\,2X^{-\frac{1}{2}}\Big|_{X_R}^{\infty} = -\frac{\sigma z}{2\epsilon_0}\left[\frac{1}{\sqrt{z^2+\infty^2}} - \frac{1}{\sqrt{z^2+R^2}}\right]$$

$$\boxed{E_z = \frac{\sigma z}{2\epsilon_0\sqrt{z^2+R^2}}}$$

Problem 4



Find $\vec{E}$ at the origin

By symmetry need only find E_z

a) $$dE_z = \frac{1}{4\pi\epsilon_0} \frac{\sigma R^2 \sin\theta \, d\theta \, d\phi}{R^2} \left(- \underbrace{\hat{R} \cdot \hat{z}}_{\cos\theta} \right)$$

$$= -\frac{\sigma}{4\pi\epsilon_0} \left(\sin\theta \, d\theta \, d\phi \right) \cos\theta$$

$$E_z = -\frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta \, d\theta$$

$$= -\frac{\sigma}{2\epsilon_0} \int_0^{\frac{\pi}{2}} -\cos\theta \, d(\cos\theta)$$

$$= \frac{\sigma}{2\epsilon_0} \left[\frac{\cos^2\theta}{2} \Big|_0^{\pi/2} \right]$$

$$= \frac{\sigma}{2\epsilon_0} \left[0 - \frac{1}{2} \right]$$

$$\boxed{E_z = -\frac{\sigma}{4\epsilon_0}}$$