Physics 412: Introduction to Electrodynamics
Homework 1
Due: Friday, 9 October, 2009

1. Problem 2.1

2. Problem 2.4

3. Problem 2.41

4. For the dipole electric field,

$$E(r,\theta) = \frac{C_\circ}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) \tag{1}$$

find and sketch the field lines.

5. Electric field of and force on continous charge distributions

a. Find the electric field for a charged sphere of radius R with spherically symmetric charge distribution. Find the field both inside and outside the sphere.

b. Place another charged sphere of radius R with an identical spherically symmetric charge distribution such that the distance between the centers of the two spheres is D, where D $>$ 2 R. Find the total force between the two charged spheres.

6. An infinitesimally thin flat disk with radius R and surface charge density $\sigma > 0$ is placed so that its center lies at the origin of a Cartesian coordinate system.

a. Find the electric field of the disk on its symmetry axis. For this problem let the z-axis be the symmetry axis of the disk.

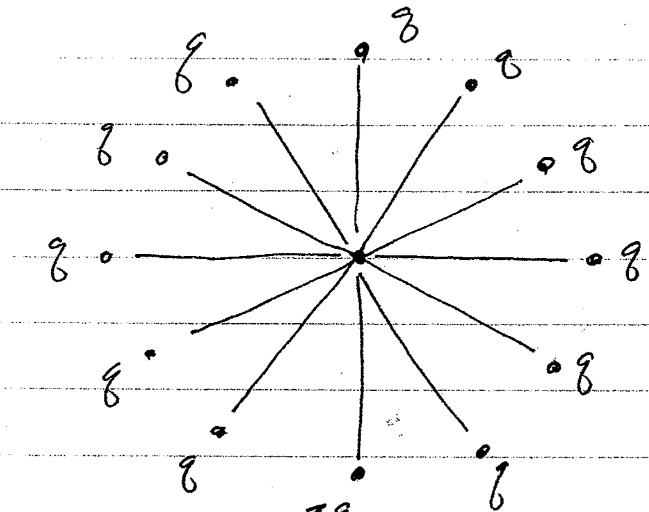
b. A charge q, where $q\sigma < 0$, is placed at the center of the disk. What is the force on charge q?

c. If charge q is constrained to move only along the z-axis find and describe its subsequent motion if it is pushed off the origin an amount $h \ll R$.

Asst #1

① Prob 2.1

a) 12 equal charges q are placed at the corners of a 12-sided polygon. What is the net force placed on a charge Q at the center of the polygon?



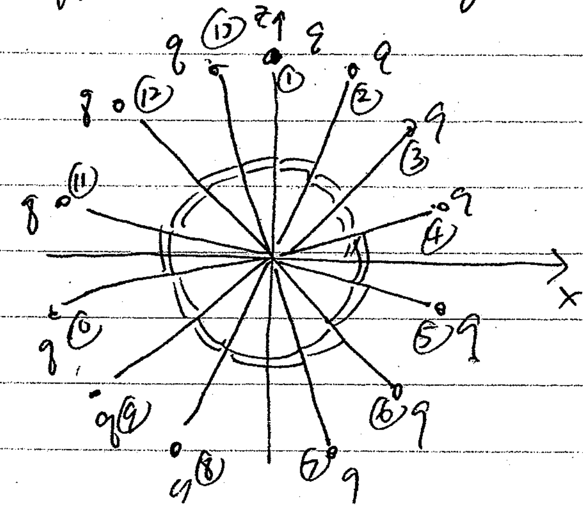
By symmetry, each charge q has an opposite partner, the net $\vec{E}$ at the center is 0

$\rightarrow \vec{F}_a = 0$

b) Suppose q is removed. What is the force on Q? If each charge is a distance R from the center, only the charge at 12 is unbalanced. If we define the z-axis as aligned from 6pm → 12 and we remove the charge at 12,

$$\rightarrow \vec{F}_a = \frac{(-1)}{4\pi\epsilon_0}\frac{qQ}{R^2}\hat{z}$$

c) Suppose we have 13 charges. What is the field at the center?



(i) Each charge is separated from its neighbor by

$$a = \frac{360°}{13}$$

ii) Each charge pair; (13, 2), (12, 3), (11, 4), (10, 5), (9, 6), (8, 7) cancels in the x-dir^{n} $\Rightarrow$ $\vec{E}_x = 0$ at the center. (charge (1) has no x-component).

$\Longrightarrow$ the only possible field can lie in the z-dir^{n}

(iii) there was nothing special about using charge 1 as the "center". This argument holds true for any other charge.

$\Longrightarrow$ $\vec{E}$ must be 0 at the origin.

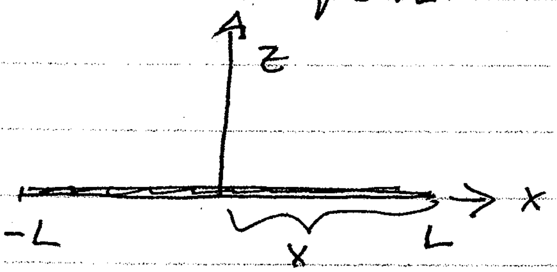
d) If charge at (1) is removed the force on Q becomes

$$\vec{F}_Q = \frac{(-1)}{4\pi\epsilon_0} \frac{qQ}{R^2} \hat{z}$$

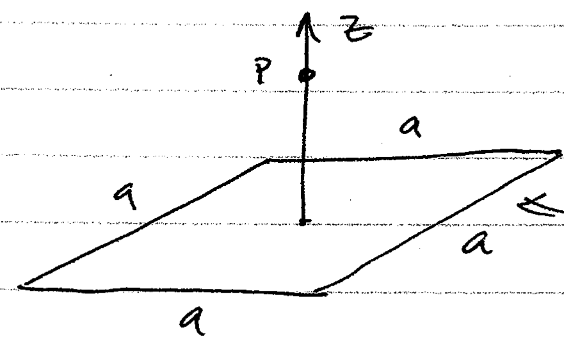
② Prob 2.4

from example 2.1, the field of a charged wire of length $2L$ and line density λ is

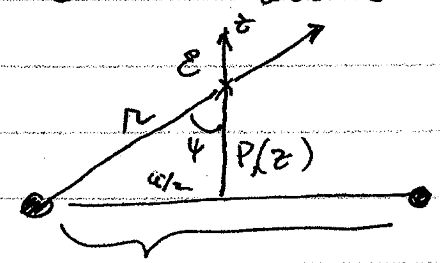
$$E_z = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2+L^2}}$$ in the z-direction



Consider a square loop w/ sides a and λ charge per unit length. Find $\vec{E}$ on the z-axis.



← from Ex 2.1, the field due to any side at P will be directed as



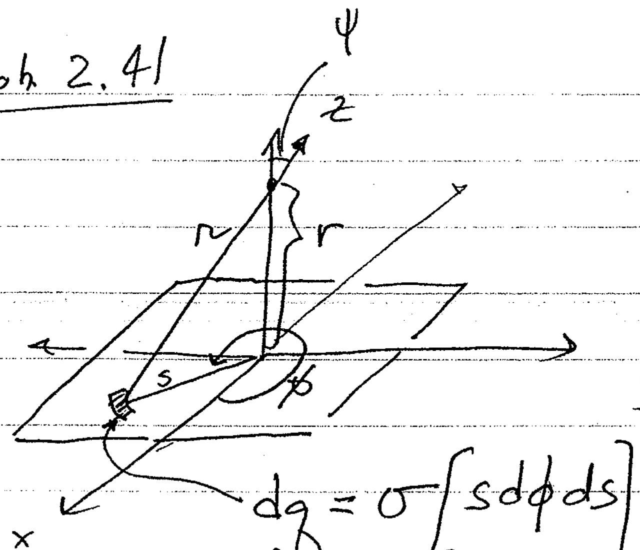
⇒ field projected on $\hat{z}$-axis is all we need. So, we want

$$\cos\psi = \frac{z}{\sqrt{\frac{a^2}{4}+z^2}}$$

$$\Rightarrow \mathcal{E}_z = \frac{1}{4\pi\epsilon_0} \frac{\lambda a}{z\sqrt{z^2+\frac{a^2}{4}}} \frac{z}{\sqrt{z^2+a^2/4}}$$

Total field is $\boxed{4\mathcal{E}_z = \frac{1}{\pi\epsilon_0} \frac{\lambda a}{(z^2+a^2/4)}}$

③ Prob. 2.41



$dq = \sigma\left[s\,d\phi\,ds\right]$ ← charge element

Use cylindrical polar coordinates

$$\rightarrow d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\sigma\, s\,ds\,d\phi}{(s^2+r^2)}$$

By symmetry the s-comp. and φ-comp. of the field are zero on the z-axis

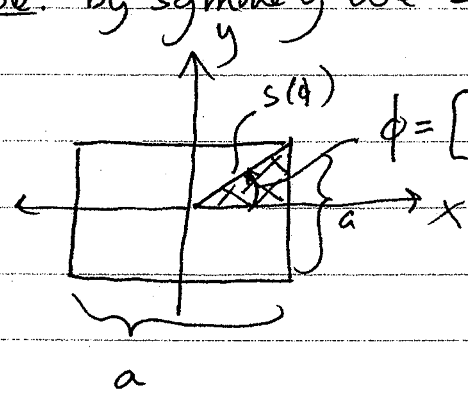
we

$$dE_z = \hat{z}\cdot d\vec{E} = \frac{\sigma}{4\pi\epsilon_0}\left[\frac{s\,ds\,d\phi}{(s^2+r^2)}\cos\psi\right]$$

$$\cos\psi = \frac{r}{\sqrt{s^2+r^2}}$$

$$\Rightarrow E_z = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{s_{max}}\frac{r\,s\,ds\,d\phi}{(s^2+r^2)^{3/2}}$$

note: by symmetry we see that we need only integrate over $\phi = [0, \frac{\pi}{4}]$ and then multiply by 8



note: we see that the limits of the s integration depend on φ as

$$s(\phi) = \frac{a}{2\cos\phi}$$

and

$$E_z = \frac{\sigma r \times 8}{4\pi\epsilon_0}\int_0^{\frac{\pi}{4}}\int_0^{\frac{a}{2\cos\phi}} \frac{s\,ds\,d\phi}{(s^2+r^2)^{3/2}}$$

to perform the s integration, let

$$W = s^2 + r^2 \longrightarrow dW = 2s\,ds$$

$$\Rightarrow E_z = \frac{2\sigma r}{\pi\epsilon_0}\int_0^{\pi/4} d\phi \int_{r^2}^{\frac{a^2}{4\cos^2\phi}+r^2} \frac{\frac{1}{2}dW}{W^{3/2}}$$

$$= \frac{\sigma r}{\pi\epsilon_0}\int_0^{\pi/4} d\phi \left[\frac{-2}{W^{1/2}}\Bigg|_{r^2}^{\frac{a^2}{4\cos^2\phi}+r^2}\right]$$

$$= -\frac{2\sigma r}{\pi\epsilon_0}\int_0^{\pi/4} d\phi \left[\frac{1}{\sqrt{\frac{a^2}{4\cos^2\phi}+r^2}} - \frac{1}{r}\right]$$

$$= -\frac{2\sigma r}{\pi\epsilon_0}\left\{\int_0^{\pi/4} d\phi \frac{\cos\phi}{r\sqrt{\frac{a^2}{4r^2}+\cos^2\phi}} - \frac{\pi}{4r}\right\}$$

$$= -\frac{2\sigma \cancel{r}}{\pi\epsilon_0 \cancel{r}}\left\{\int_0^{\pi/4} \frac{d(\sin\phi)}{\sqrt{\frac{a^2}{4r^2}+1-\sin^2\phi}} - \frac{\pi}{4}\right\}$$

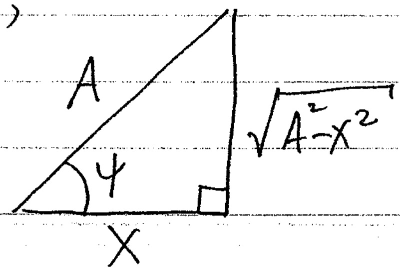
do the integral

let $A^2 = \left(1+\frac{a^2}{4r^2}\right)$; $X = \sin\phi \rightarrow dX = d(\sin\phi)$

$$E_z = \frac{\sigma}{2\epsilon_0} - \frac{2\sigma}{\pi\epsilon_0}\int \frac{dX}{\sqrt{A^2 - X^2}}$$

w/appropriate limits

Consider,



$$\Longrightarrow \begin{cases} \tan\psi = \sqrt{\dfrac{A^2}{X^2}-1} \\[2mm] \dfrac{A}{\sqrt{A^2-X^2}} = \dfrac{1}{\sin\psi} \\[2mm] \cos\psi = \dfrac{X}{A} \\[2mm] \quad \longrightarrow -\sin\psi\, d\psi = \dfrac{dX}{A} \end{cases}$$

substitute into the integral,

$$E_z = \frac{\sigma}{2\epsilon_0} - \frac{2\sigma}{\pi\epsilon_0}\int -A\sin\psi\, d\psi \times \left(\frac{1}{A\sin\psi}\right)$$

$$= \frac{\sigma}{2\epsilon_0} + \frac{2\sigma}{\pi\epsilon_0}\left[\psi\Big|_{\psi_0}^{\psi_1}\right]$$

$$= \frac{\sigma}{2\epsilon_0} + \frac{2\sigma}{\pi\epsilon_0}\left[\tan^{-1}\left\{\sqrt{\frac{1+a^2/4r^2}{\sin^2\phi}-1}\right\}\right]_0^{\pi/4}$$

$$= \frac{\sigma}{2\epsilon_0} + \frac{2\sigma}{\pi\epsilon_0}\left[\tan^{-1}\sqrt{2+\frac{a^2}{2r^2}-1} - \tan^{-1}\sqrt{\infty}\right]$$

$$= \frac{\sigma}{2\epsilon_0} + \frac{2\sigma}{\pi\epsilon_0}\left[\tan^{-1}\sqrt{1+\frac{a^2}{2r^2}} - \frac{\pi}{2}\right]$$

$$\boxed{E_z = \frac{\sigma}{2\epsilon_0}\left[\frac{4}{\pi}\tan^{-1}\sqrt{1+\frac{a^2}{2r^2}} - 1\right]}$$ ✓

infinite charged sheet

a) $a \to \infty \Longrightarrow E_z \to \frac{\sigma}{2\epsilon_0}\left[\frac{4}{\pi}\left(\frac{\pi}{2}\right)-1\right] = \frac{\sigma}{2\epsilon_0}$ ✓

b) $z \gg a \Longrightarrow E_z \to \frac{\sigma}{2\epsilon_0}\left[\ ?,\ \text{what is the expansion for } \tan^{-1}\ ?\right]$

looked it up and find, $E_z \rightarrow \left[\frac{4\sigma}{\pi 2\epsilon_0}\left(\frac{\pi}{4} + \frac{a^2}{8r^2}\right) - \frac{\sigma}{2\epsilon_0}\right]$

$$= \frac{\sigma a^2}{4\pi\epsilon_0 r^2} \quad \sigma a^2 \equiv Q$$

$$\longrightarrow E_z = \frac{Q}{4\pi\epsilon_0 r^2} \checkmark$$

④ $\vec{E} = \frac{c_0}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$

a) Find and sketch the field lines

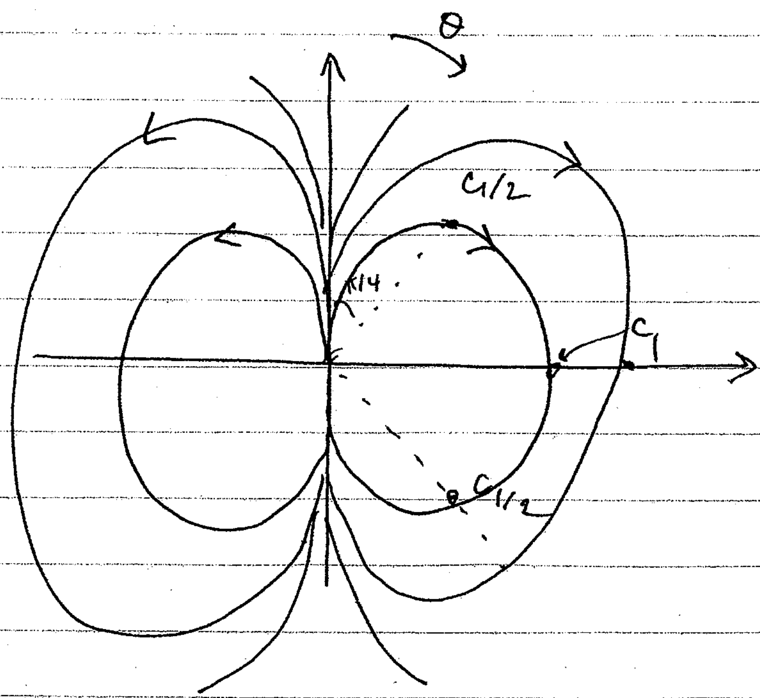
$$\frac{1}{r}\frac{dr}{d\theta} = \frac{E_r}{E_\theta} \longrightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{2\cos\theta}{\sin\theta}$$

$$\text{and} \int \frac{dr}{r} = 2\int \frac{\cos\theta\, d\theta}{\sin\theta}$$

$$\ln r = 2\int \frac{d(+\sin\theta)}{\sin\theta}$$

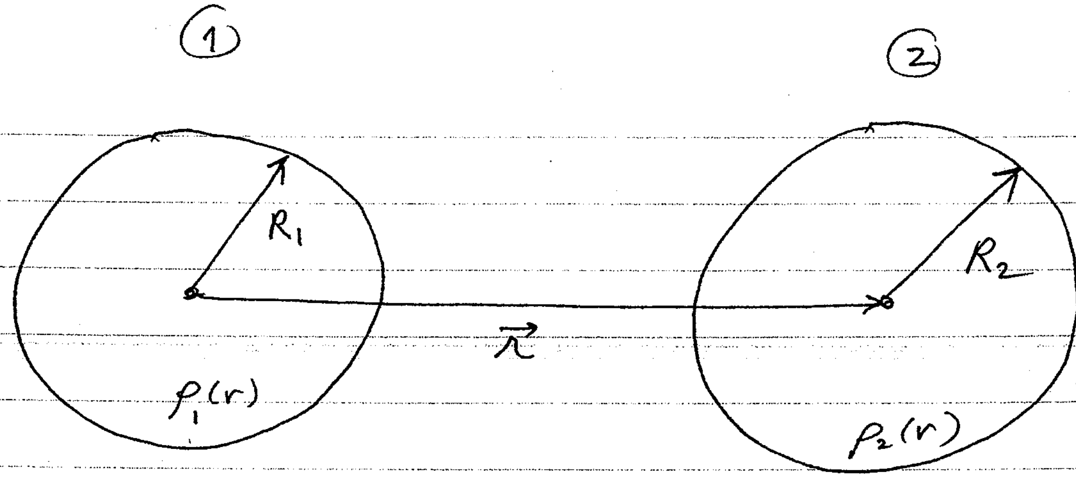
$$= +2\ln\sin\theta + C$$

$$\boxed{r = C_1 \sin^{+2}\theta}$$; defines a family of sol$^{\text{ns}}$ parametized by C_1



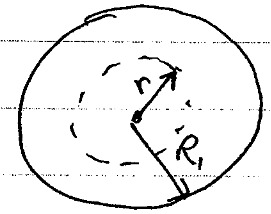
θ	r
0	0
π/4	$C_1/2$
π/2	C_1
3π/4	$C_1/2$
π	0

5



Find force of sphere 1 on sphere 2

(a) Find field of sphere 1. Use Gauss's law because of symmetry



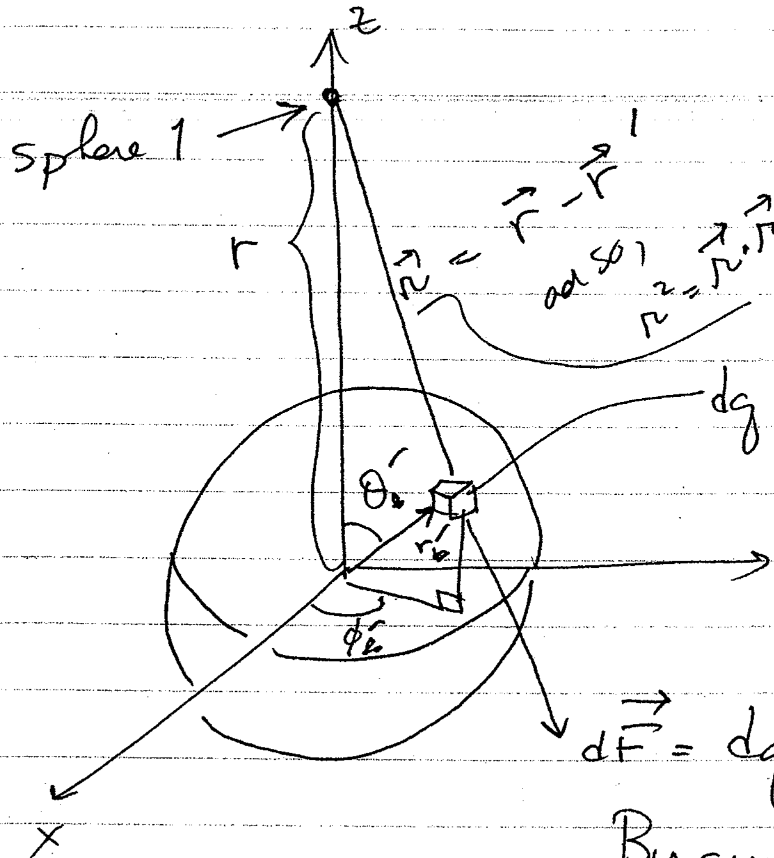
$$\underbrace{\oint \vec{E}\cdot d\vec{S}}_{\text{on sphere w/radius } r} = \underbrace{\frac{\int \rho\, d^3x}{\epsilon_0}}_{\text{over sphere w/radius } r}$$

$$E_r 4\pi r^2 = \frac{\overbrace{Q(r)}^{\text{charge contained in sphere of radius } r}}{\epsilon_0}$$

and for $r > R_1 \Longrightarrow \boxed{E_r = \dfrac{Q_1}{4\pi\epsilon_0 r^2}}$

the field acts as if all of the charge were concentrated at the center of the sphere.

(b) Find force of sphere 2



$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ and so

$\mathscr{r}^2 = \vec{\mathscr{r}} \cdot \vec{\mathscr{r}} = r^2 + r'^2 - 2rr'\cos\theta'$

$dq' = \rho_2 r'^2 \sin\theta' d\theta' d\phi' dr'$

$d\vec{F} = dq'\vec{E}_1 \equiv$ force exerted on dq

By symmetry, we need only consider the force along z.

$$\Longrightarrow dF_z = \hat{z}' \cdot d\vec{F}' = (dq'E_1)\left[\frac{r - r'\cos\theta'}{|\vec{\mathscr{r}}|}\right]$$

$$F_z = \int \rho_2 r'^2 dr' \sin\theta' d\theta' d\phi' \left[\frac{Q_1}{4\pi\epsilon_0}\left(\frac{r - r'\cos\theta'}{\{r^2 + r'^2 - 2rr'\cos\theta'\}^{3/2}}\right)\right]$$

$$\begin{cases} \underline{\text{note}}: \int d\phi' = 2\pi . \\ \underline{\text{note}}: \sin\theta d\theta = d(-\cos\theta) \end{cases}$$

$$\Longrightarrow F_z = \frac{-Q_1}{2\epsilon_0}\int \rho_2 r'^2 dr' \left[\frac{r - r'\cos\theta'}{(r^2 + r'^2 - 2rr'\cos\theta')^{3/2}}\right] d(\cos\theta')$$

<u>let:</u> $W = r^2 + r'^2 - 2rr'\cos\theta'$

$$dW = -2rr'\,d(\cos\theta')$$

$$\rightarrow = \frac{Q_1}{2\epsilon_0}\int \rho_2 r'^2 dr' \left[\frac{r - r'\left(\frac{r^2+r'^2-W}{2rr'}\right)}{W^{3/2}}\right]\frac{dW}{2rr'}$$

$$= \frac{Q_1}{4\epsilon_0 r}\int \rho_2 r' dr' \left[\frac{\overbrace{r - \frac{1}{2}\frac{r^2+r'^2}{r}}^{\frac{1}{2r}(r^2-r'^2)}}{W^{3/2}} + \frac{1}{2r}\frac{1}{W^{1/2}}\right] dW$$

$$= \frac{Q_1}{4\epsilon_0 r}\int \rho_2 r' dr' \left[\frac{(r^2-r'^2)}{\cancel{2}r}\frac{-\cancel{2}}{W^{1/2}} + \frac{1}{\cancel{2}r}\cancel{2}W^{1/2}\right]_{W_0=(r-r')^2}^{W_1=(r+r')^2}$$

$$= \frac{Q_1}{4\epsilon_0 r^2}\int \rho_2 r' dr' \left[-(r^2-r'^2)\left[\frac{1}{(r+r')} - \frac{1}{(r-r')}\right] + (r+r') - (r-r')\right]$$

$$= \frac{Q}{4\epsilon_0 r^2}\int \rho_2 r' dr' \left[\underbrace{-(r-r') + (r+r') + 2r'}_{4r'}\right]$$

$$= \frac{Q_1}{4\epsilon_0 r^2}\int 4\rho_2 r'^2 dr'$$

$$= \frac{Q_1}{\epsilon_0 r^2}\int \rho_2 r'^2 dr'$$

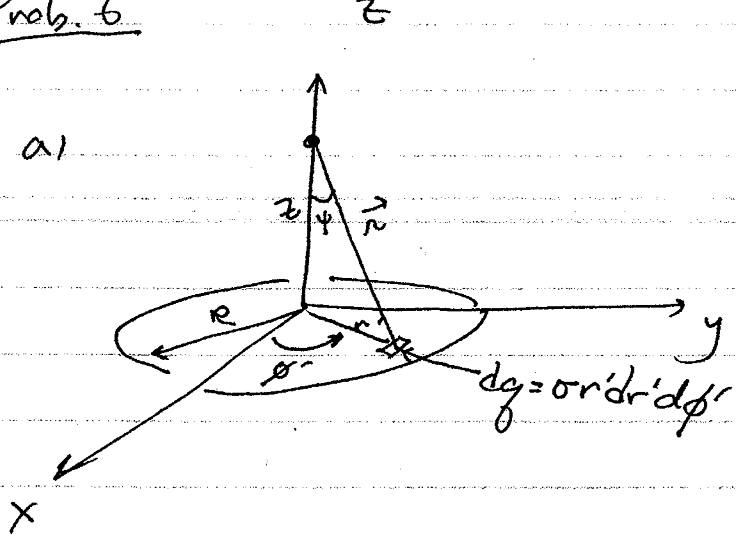
In spherical symmetry, this integral is $\frac{Q_2}{4\pi}$

$$\rightarrow \boxed{F_z = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^2}}$$

; behaves as if both spheres were point charges!

Prob. 6

a)



By symmetry, we need only calculate the z-compont for dq

$$\Rightarrow dE_z = \hat{z}\cdot d\vec{E}_q = \frac{\hat{z}\cdot}{4\pi\epsilon_0}\left(\frac{\sigma r' dr' d\phi'}{r^2}\right)$$

$$= \frac{\sigma}{4\pi\epsilon_0}\left[\frac{r' dr' d\phi'}{r^2} \times \frac{z}{r}\right]$$

$$\Rightarrow E_z = \frac{\sigma z}{4\pi\epsilon_0}\int_0^R dr' \int_0^{2\pi} d\phi' \frac{r'}{(r'^2+z^2)^{3/2}}$$

$$= \frac{\sigma z}{2\epsilon_0}\int_0^R \frac{r' dr'}{(r'^2+z^2)^{3/2}}$$

Let $X = r'^2 + z^2 \longrightarrow dX = 2r'dr' \longrightarrow r'dr' = \frac{dX}{2}$

$$= \frac{\sigma z}{2\epsilon_0}\int_{z^2}^{R^2+z^2} \frac{\frac{1}{2}dX}{X^{3/2}}$$

$$= \frac{\sigma z}{4\epsilon_0}\left(\frac{-2}{X^{1/2}}\right)\Bigg|_{z^2}^{R^2+z^2}$$

$$\boxed{E_z = -\frac{\sigma z}{2\epsilon_0}\left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{|z|}\right]}$$

b) at $x = y = z = 0 \Rightarrow E_z = 0 \rightarrow F_q = 0$

c) find E_z for $|h/R| \ll 1$ or $(|z| \ll R)$

$$E_z = -\frac{\sigma z}{2\epsilon_0}\left[\frac{1}{R}\frac{1}{\sqrt{1+z^2/R^2}} - \frac{1}{|z|}\right]$$

$$\approx -\frac{\sigma z}{2\epsilon_0}\left[\frac{1}{R}\left(1-\frac{1}{2}\frac{z^2}{R^2}\right) - \frac{1}{|z|}\right]$$

$$\approx \frac{\sigma}{2\epsilon_0}\left[\frac{z}{|z|} - \left(\frac{z}{R}\right)\right]$$

<u>Equation-of-Motion is</u>

$$m\ddot{z} = \frac{\overbrace{q\sigma}^{<0}}{2\epsilon_0}\left[\frac{z}{|z|} - \left(\frac{z}{R}\right)\right] \simeq \frac{q\sigma}{2\epsilon_0}\left[\frac{z}{|z|}\right]$$

(i) if the charge is released from $z_0 = h$; at rest, $\dot{z}_0 = 0$

$$\Rightarrow \begin{cases} z = \frac{1}{2}\frac{q\sigma}{2\epsilon_0}\overbrace{\left[\frac{z}{|z|}\right]}^{=+1}t^2 + h \\ \dot{z} = \frac{q\sigma}{2\epsilon_0}\left[\frac{z}{|z|}\right]t \end{cases}$$

the charge falls to disk, reaching the disk after time

$$t_f = \sqrt{-\frac{4\epsilon_0 h}{q\sigma}}$$

if a hole is drilled in the disk to allow the charge to pass through the disk, then it move to a distance $+\underline{h}$ below the disk and stop and then fall back toward the disk, passing through it and returning to height $\underline{h}$. The motion is periodic w/ period, $P = 4t_f$.