Physics 412: Introduction to Electrodynamics
Homework 2
Due: Friday, 16 October, 2009

7. Problem 2.9

8. Problem 2.14

9. Problem 2.17

10. Problem 2.18

11. Problem 2.19

12. Problem 2.20

(9) Prob. 2.9

Given $\vec{E} = kr^3\hat{r}$, find

a) $\rho(r)$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2E_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^5\right) = \rho/\epsilon_0$$

$$5kr^2 = \rho/\epsilon_0$$

$$\boxed{\rho(r) = 5k\epsilon_0 r^2}$$

b) $Q(R)$

(i) $$Q(R) = \int_0^R \rho(r)\,4\pi r^2 dr$$

$$= 5k\epsilon_0\int_0^R 4\pi r^4 dr$$

$$\boxed{Q(R) = k\epsilon_0 4\pi R^5}$$

(ii) $$\oint\vec{E}\cdot d\vec{a} = \frac{Q}{\epsilon_0}$$

$$kR^3 4\pi R^2 = \frac{Q}{\epsilon_0}$$

$$4\pi kR^5 = Q/\epsilon_0$$

$$\rightarrow \boxed{Q = 4\pi k\epsilon_0 R^5}$$

(10) Prob 2.14

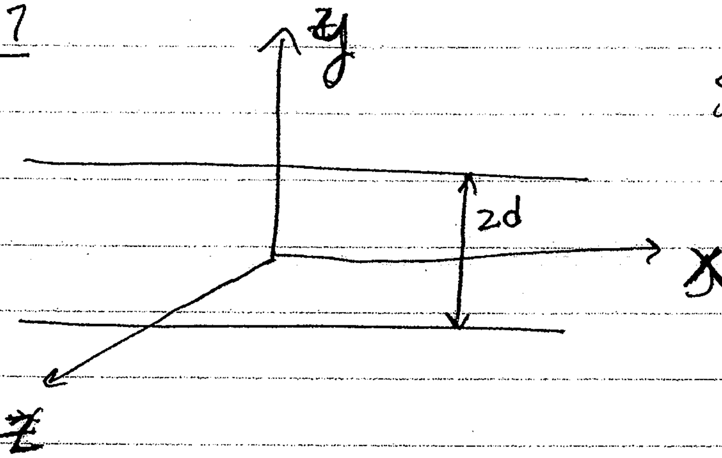
Find $\vec{E}$ if $\rho = kr$.

$$\oint_{\vec{F}} \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0}\int_0^r (kr)\, 4\pi r^2 dr$$

$$E_r 4\pi r^2 = \frac{4\pi k}{\epsilon_0}\frac{r^4}{4}$$

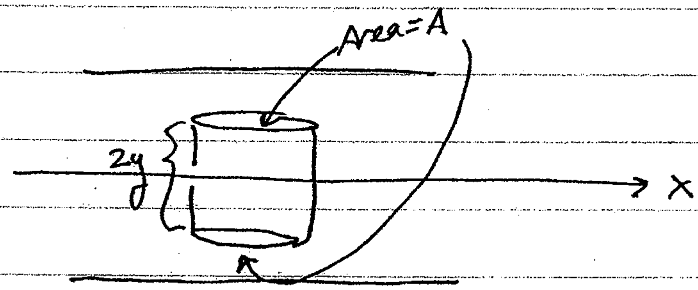
$$\rightarrow \boxed{E_r = \frac{k}{4\epsilon_0} r^2, \quad r < R = \text{radius of sphere}}$$

2.17



Slab has uniform charge density ρ. We see from symmetry arguments that there will only be a y-component for the field. Let's use Gauss's law to find E_y.

a) $|y| < d$ ← inside the slab



$$\oint \vec{E} \cdot d\vec{S} = \int \frac{\rho \, d\tau}{\epsilon_0}$$

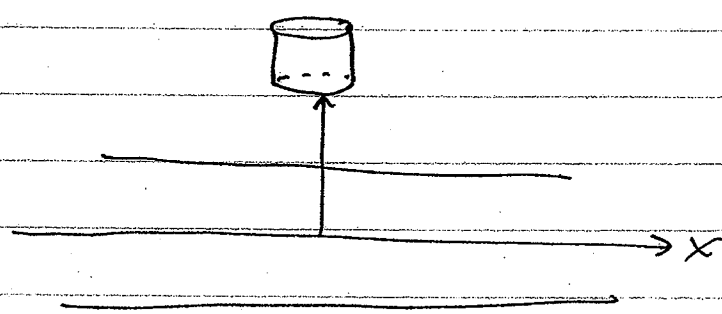
$$E_y^+ A - E_y^- A = 2Ay \frac{\rho}{\epsilon_0}$$

are in opposite directions

$$2E_y = 2y\rho / \epsilon_0$$

$$\rightarrow \boxed{\vec{E}_y = \frac{\rho y}{\epsilon_0} \hat{y}}$$

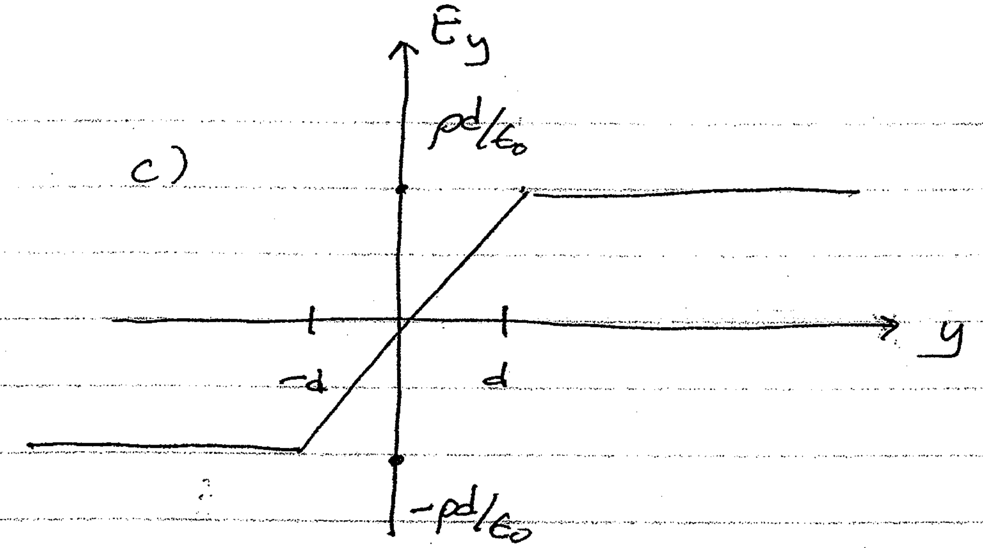
b) Outside slab



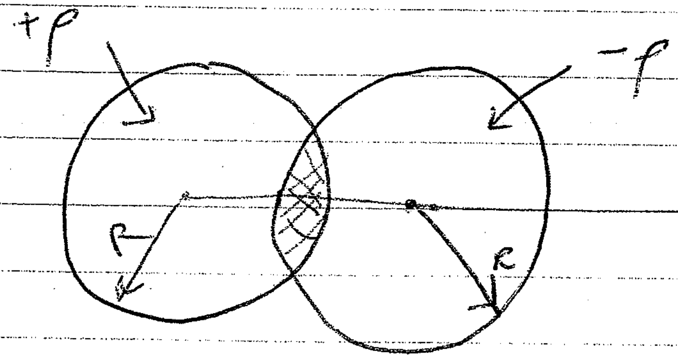
$$E_y^+ A - E_y^- A = 0$$

$$\rightarrow E_y^+ = E_y^-$$

E_y is constant above & below the slab c)



(12) Prob 2.18



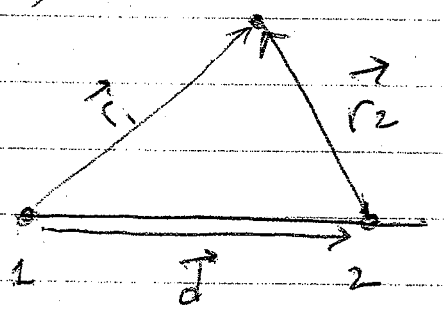
find $\vec{E}$ in the overlap region

(a) the $\vec{E}$ of a uniformly charged sphere is

$$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0}\int \rho \, d^3x$$

$$E_r \, 4\pi r^2 = \frac{1}{\epsilon_0} 4\pi \rho_0 \frac{r^3}{3} \rightarrow \boxed{\vec{E}_r = \frac{\rho_0}{3\epsilon_0} r \hat{r}}$$

(b) clearly



and $\vec{E}_{Total} = \frac{\rho_0}{3\epsilon_0}\vec{r}_1 - \frac{\rho_0}{3\epsilon_0}\vec{r}_2$

$$= \frac{\rho}{3\epsilon_0}(\vec{r}_1 - \vec{r}_2)$$

$$\boxed{\vec{E}_{Tot} = \frac{\rho}{3\epsilon_0}\vec{d}}$$

2.19

Calculate $\vec{\nabla} \times \vec{E}$ from $\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')d\tau'}{|\vec{r}-\vec{r}\,'|^3}(\vec{r}-\vec{r}\,')$

$$\Rightarrow \vec{\nabla}_r \times \vec{E} = \frac{1}{4\pi\epsilon_0}\vec{\nabla}_r \times \int \rho(\vec{r}\,')d\tau'\left[\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right]$$

to keep it separate from $\vec{r}\,'$, the integration variable

a) does not operate on $\vec{r}\,'$ $\Rightarrow$ can be dragged into the integral and moved past $\rho(\vec{r}\,')d\tau'$, if desired. Can do this and find the answer

b) however, let's do it another way. look at $\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$. Note that

$$\vec{\nabla}_r\left[\frac{1}{|\vec{r}-\vec{r}\,'|}\right] = \vec{\nabla}_r\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right]$$

$$= \hat{x}\left[\frac{-(x-x')}{|\vec{r}-\vec{r}\,'|^3}\right] + \hat{y}\left[\frac{-(y-y')}{|\vec{r}-\vec{r}\,'|^3}\right] + \hat{z}\left[\frac{-(z-z')}{|\vec{r}-\vec{r}\,'|^3}\right]$$

$$= -\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$\Rightarrow \vec{\nabla}_r \times \vec{E} = \frac{1}{4\pi\epsilon_0}\vec{\nabla}_r \times \int \rho(\vec{r}\,')d\tau'\left[-\vec{\nabla}_r\frac{1}{|\vec{r}-\vec{r}\,'|}\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\vec{\nabla}_r \times \vec{\nabla}_r \underbrace{\int\left\{-\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right\}d\tau'}_{\text{scalar function}}$$

$$\Rightarrow \boxed{\vec{\nabla}_r \times \vec{E} = 0}$$

2.20

(I) Which field is electrostatic?

a) $\vec{E} = k\left[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}\right]$

$\vec{\nabla}\times\vec{E} = k(0-2y,\ 0-3z,\ 0-x) \neq 0$

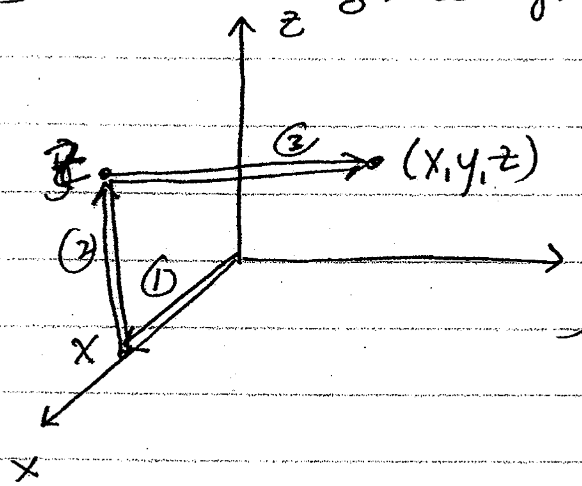
$\rightarrow$ not electrostatic

b) $\vec{E} = k\left(y^2\hat{x} + \left[2xy+z^2\right]\hat{y} + 2yz\,\hat{z}\right)$

$\vec{\nabla}\times\vec{E} = k(2z-2z,\ 0-0,\ 2y-2y) = 0$

$\rightarrow$ electrostatic

(II) Find V using the origin as the reference point



Take the 3 paths as shown to the left:

① $x'=[0,x],\ y'=0,\ z'=0$

② $x'=x,\ y'=0,\ z'=[0,z]$

③ $x'=x,\ y'=[0,y],\ z'=z$

$$-\int dV = \int_0^x \cancel{ky^2}^{\,0}\,dx + \int_0^z \cancel{2kyz}^{\,0}\,dz + \int_0^y k(2xy+z^2)\,dy = (xy^2+yz^2)k$$

$$\Rightarrow -V(x,y,z) + V(0,0,0) = (xy^2+yz^2)k$$

$$\boxed{V(x,y,z) = \underbrace{V(0,0,0)}_{\text{set to } 0} - k(xy^2+yz^2)}$$

(III) Find $\vec{E} = -\vec{\nabla} V$

$$= k\left[-y^2\hat{x} - (2xy + z^2)\hat{y} - 2zy\,\hat{z}\right]$$

$$\boxed{\vec{E} = k\left(y^2\hat{x} + (2xy + z^2)\hat{y} + 2zy\,\hat{z}\right)}$$