Physics 412: Introduction to Electrodynamics
Homework 2
Due: Not to be handed in. Test 1 is on Friday, October 23, 2009

13. Problem 2.22

14. Problem 2.26

15. Problem 2.32

16. Problem 2.35

17. Problem 2.36

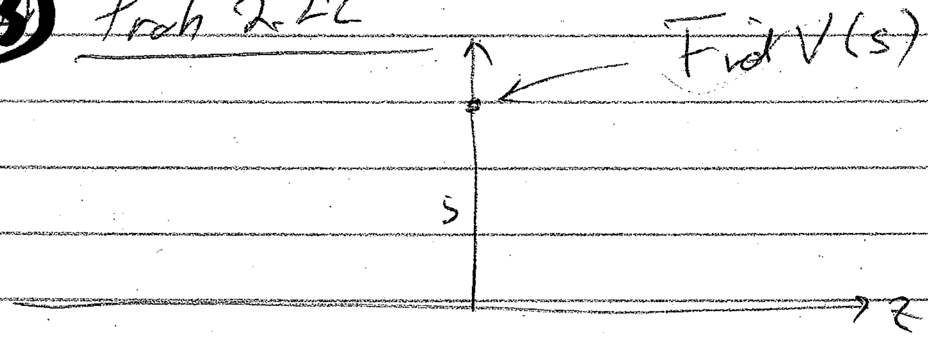
18. The Yukawa Potential is given by

$$V = \kappa \frac{e^{-\alpha r}}{r} \tag{1}$$

where κ and α are constants. Find the charge distribution which produces this field. What is the total charge? How much energy is needed to *unbind* the system?

Asst #3

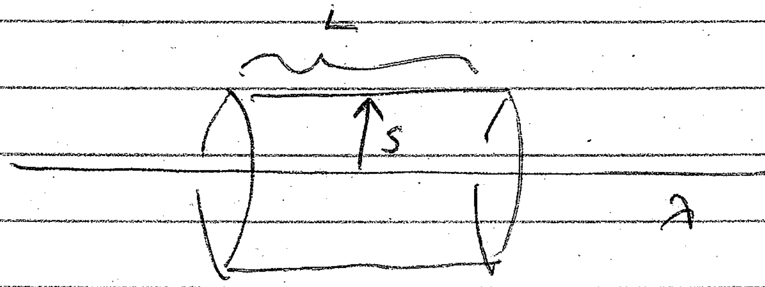
(B) Prob 2.22



Sol^{n}

a) find $dV = -\vec{E} \cdot d\vec{r}$

b) By symmetry $\vec{E}$ has only a radial component
→ use Gauss's law



$$\oint \vec{E} \cdot d\vec{S} = E_s 2\pi s L = \frac{\lambda L}{\epsilon_0} \rightarrow \boxed{\vec{E}_s = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}}$$

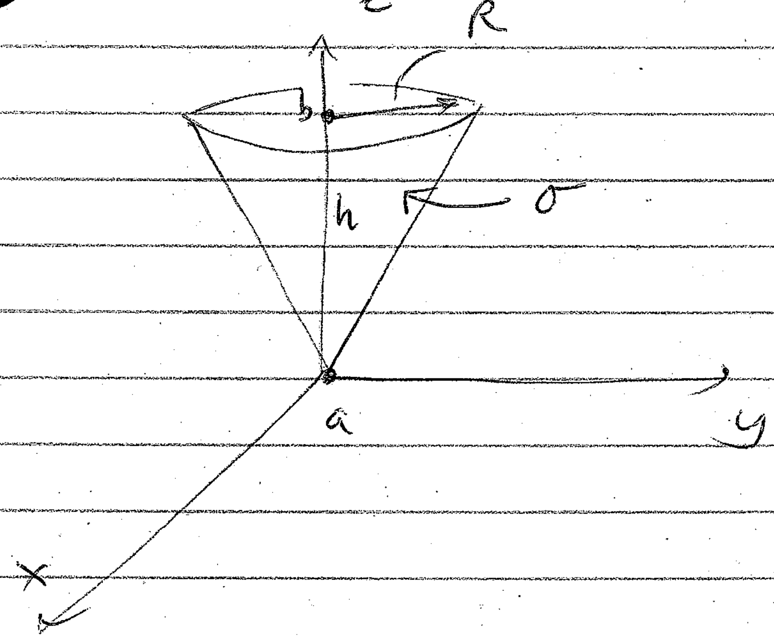
c) $$V = -\int \vec{E}_s \cdot d\vec{S} = -\int_{s_0}^{s} \frac{\lambda}{2\pi\epsilon_0}\frac{ds}{s}$$

$$\boxed{V - V_0 = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s}{s_0}\right)}$$

integration limit

d) $$\vec{E} = -\vec{\nabla} V = +\frac{\lambda}{2\pi\epsilon_0}\frac{\hat{s}}{s}$$ ✓

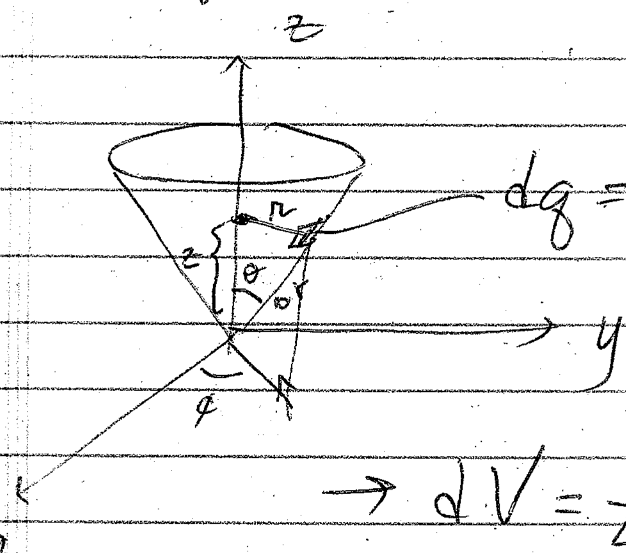
(14) Prob. 2.26



Find ΔV between b and a

Sol^{n}

Find V on the z-axis; Use spherical coordinates



$dq = \sigma r \, d\phi \, dr$

$$\rightarrow dV = \frac{1}{4\pi\epsilon_0} \frac{\sigma r \, d\phi \, dr}{\sqrt{r^2 + z^2 - 2zr\cos\theta}}$$

$$V = \frac{\sigma}{4\pi\epsilon_0} \int \frac{r \, dr \, d\phi}{\sqrt{r^2 + z^2 - 2zr\cos\theta}}$$

Integrate $\int d\phi$

$$V = \frac{\sigma}{2\epsilon_0}\int_0^{r_0} \frac{r\,dr}{\sqrt{r^2+z^2-2zr\cos\theta}}$$

$$= \frac{\sigma}{2\epsilon_0}\Big[\sqrt{r^2-2zr\cos\theta+z^2} + z\cos\theta \ln\left\{-z\cos\theta + r + \sqrt{r^2-2zr\cos\theta+z^2}\right\}\Big]_0^{r_0}$$

okay; integration limits are 0 and r_0, where

$$r_0 = \sqrt{h^2+R^2}$$

okay; what are $\cos\theta$, ... ?

$$\cos\theta = \frac{h}{\sqrt{h^2+R^2}} = \frac{h}{r_0} \Rightarrow r_0 = \frac{h}{\cos\theta}$$

Evaluate integral

$$V = \frac{\sigma}{2\epsilon_0}\Big[\sqrt{h^2+R^2-2zh+z^2} + \frac{zh}{\sqrt{h^2+R^2}}\ln\left\{-\frac{zh}{\sqrt{h^2+R^2}} + \sqrt{h^2+R^2} + \sqrt{h^2+R^2-2zh+z^2}\right\} - \left\{z + \frac{zh}{\sqrt{h^2+R^2}}\ln\left(-\frac{zh}{\sqrt{R^2+h^2}} + z\right)\right\}\Big]$$

find $\Delta V = V(h) - V(0)$

a) $$V(0) = \frac{\sigma}{2\epsilon_0}\left[\left(\sqrt{h^2+R^2}+0\right) - (0)\right]$$

$$= \frac{\sigma\sqrt{h^2+R^2}}{2\epsilon_0}$$

b) $$V(h) = \frac{\sigma}{2\epsilon_0}\left[R + \frac{h^2}{\sqrt{h^2+R^2}}\ln\left\{\frac{h^2}{\sqrt{h^2+R^2}} + \sqrt{h^2+R^2}+R\right\}\right.$$

$$\left. - \left\{h + \frac{h^2}{\sqrt{h^2+R^2}}\ln\left\{h - \frac{h^2}{\sqrt{R^2+h^2}}\right\}\right\}\right]$$

$$= \frac{\sigma}{2\epsilon_0}\left[(R-h) + \frac{h^2}{\sqrt{h^2+R^2}}\ln\left\{\frac{h^2+h^2+R^2+R\sqrt{h^2+R^2}}{h\sqrt{R^2+h^2}-h^2}\right\}\right]$$

$$= \frac{\sigma}{2\epsilon_0}\left[(R-h) + \frac{h^2}{\sqrt{h^2+R^2}}\ln\left\{\frac{2h^2+R^2+R\sqrt{h^2+R^2}}{h\sqrt{R^2+h^2}-h^2}\right\}\right]$$

$$\Rightarrow \Delta V = \frac{\sigma}{2\epsilon_0}\left[(R-h) + \frac{h^2}{\sqrt{h^2+R^2}}\ln\left\{\frac{2h^2+R^2+R\sqrt{h^2+R^2}}{h\sqrt{R^2+h^2}-h^2}\right\} - \sqrt{h^2+R^2}\right]$$

<u>for the case $h = R$</u>

$$\Delta V = \frac{\sigma R}{2\sqrt{2}\epsilon_0}\left[\ln\left(\frac{\sqrt{2}+3}{\sqrt{2}-1}\right) - \frac{1}{2}\right]$$

Problem 2.32

Find the energy stored in a uniformly charged solid sphere of radius R and charge q. Do 3 different ways:

ⓐ Eq 2.43

ⓑ Eq 2.45

ⓒ Eq 2.44. Take a spherical shell of radius a. What happens as $a \to \infty$?

ⓐ

$$W = \frac{1}{2}\int \rho V \, d\tau \qquad (2.43)$$

(i) find V from $\rho = \begin{cases} \frac{3q}{4\pi R^3} & , r < R \\ 0 & , r > R \end{cases}$

from Gauss's law find $\vec{E}$. Need only the radial component from symmetry. For $r < R$

$$\Rightarrow E_r 4\pi r^2 = \frac{1}{\epsilon_0}\left[\frac{3q}{4\pi R^3}\right]\frac{4\pi}{3} r^3 = \frac{q}{\epsilon_0}\left(\frac{r}{R}\right)^3$$

$$\Rightarrow E_r = \frac{q}{4\pi\epsilon_0 R^3} r \quad , \quad r < R$$

and so, $\cancel{-dV = \vec{E}\cdot d\vec{r} = \frac{1}{4\pi\epsilon_0 R}}$

For $r > R$

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \quad , r > R$$

$$\Rightarrow -\int_\infty^r dV = \int_\infty^r \vec{E}\cdot d\vec{r}$$

$$-V(r)+V_\infty = \int_\infty^R \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr + \int_R^r \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\,r\,dr$$

$$= -\frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \cancelto{0}{\frac{1}{\infty}}\right) + \frac{q}{4\pi\epsilon_0 R^3}\left(\frac{r^2}{2}-\frac{R^2}{2}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{r^2}{2R^3}-\frac{3}{2R}\right)$$

set $V_\infty = 0$

$$\Rightarrow V(r) = \frac{q}{8\pi\epsilon_0 R}\left(3-\frac{r^2}{R^2}\right), \quad r<R$$

$$\text{find } W = \frac{1}{2}\int_0^R \rho\,\underbrace{\left[\frac{q}{8\pi\epsilon_0 R}\left(3-\frac{r^2}{R^2}\right)\right]}_{\substack{\text{we need only consider } r<R \\ \text{because } \rho=0 \text{ for } r>R}} 4\pi r^2\,dr$$

$$= \frac{1}{2}\left(\frac{3q}{4\pi R^3}\right)\left(\frac{q}{2\epsilon_0 R}\right)\left[R^3-\frac{R^3}{5}\right]$$

$$\boxed{W = \frac{3q^2}{20\pi\epsilon_0 R}}$$

(b) $W = \frac{\epsilon_0}{2}\int E^2 d\tau$

$$= \frac{\epsilon_0}{2}\int_0^R \left(\frac{qr}{4\pi\epsilon_0 R^3}\right)^2 4\pi r^2 dr + \frac{\epsilon_0}{2}\int_R^\infty \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$= \frac{q^2}{4\pi\epsilon_0 \times 2}\int_0^R \frac{r^4 dr}{R^6} + \frac{q^2}{4\pi\epsilon_0 \times 2}\int_R^\infty \frac{dr}{r^2}$$

$$= \frac{q^2}{8\pi\epsilon_0}\left[\frac{R^5}{5R^6} - \left(\cancel{\frac{1}{\infty}}^{\,0} - \frac{1}{R}\right)\right]$$

$$\boxed{W = \frac{3q^2}{20\pi\epsilon_0 R}}$$

(c) $W = \frac{\epsilon_0}{2}\left[\int_0^a E^2 d\tau + \oint V\vec{E}\cdot d\vec{S}\right]$

$$= \frac{q^2}{8\pi\epsilon_0}\left[\int_0^R \frac{r^4 dr}{R^6} + \int_R^a \frac{dr}{r^2}\right] + \frac{\epsilon_0}{2}\oint \frac{q}{4\pi\epsilon_0 a}\frac{qa^2 d\Omega}{4\pi\epsilon_0 a}$$

$$= \frac{q^2}{8\pi\epsilon_0}\left[\frac{1}{R} - \left(\frac{1}{a} - \frac{1}{R}\right)\right] + \frac{q^2}{8\pi\epsilon_0 a}$$

$$\boxed{W = \frac{3q^2}{20\pi\epsilon_0 R}, \text{ independent of } a}$$

Problem 2.35

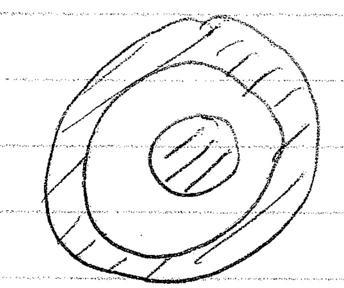
a metal sphere of radius R, carrying charge q is surrounded by a thick concentric shell (inner radius a, outer radius b). The shell carries no net charge.

(a) Find $\sigma(R)$, $\sigma(a)$, $\sigma(b)$

(b) Find $V(0)$ using V_∞ as the reference point

(c) Ground b which lowers its potential to $V=0$. How do your answers to a & b change?

Sol^n



$$\Rightarrow E_r = \begin{cases} 0 & , r<R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} & , R<r<a \\ 0 & , a<r<b \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} & , b<r \end{cases}$$

if ungrounded

(a) $\sigma(R) = \frac{q}{4\pi R^2}$, $\sigma(a) = -\frac{q}{4\pi a^2}$, $\sigma(b) = \frac{q}{4\pi b^2}$

(b) $-\int_\infty^0 dV = \int_\infty^b \frac{q\,dr}{4\pi\epsilon_0 r^2} + \int_a^R \frac{q\,dr}{4\pi\epsilon_0 r^2}$ + other terms which are 0, because local E_r is 0

$$= \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{b} - \frac{1}{R} + \frac{1}{a}\right]$$

$$\Rightarrow V(0) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{b} + \frac{1}{R} - \frac{1}{a}\right]$$

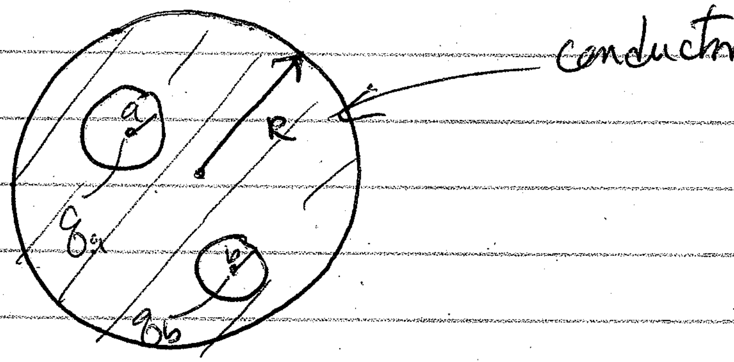
(c) Grnd b $\Rightarrow$ $V(b)=0 \Rightarrow \sigma(b)=0$
and $\sigma(R)$ and $\sigma(a)$ are unchanged

(c') $$-\int_{\infty}^{0} dV = \int_{a}^{R} \frac{q\,dr}{4\pi\epsilon_0 r^2}$$ + other terms which are zero because $E_r=0$

$$= -\frac{q}{4\pi\epsilon_0}\left[\frac{1}{R}-\frac{1}{a}\right]$$

$$V(0) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R}-\frac{1}{a}\right]$$

11 Prob 2.36



a) Find σ_a, σ_b, and σ_R

(i) to cancel q_a, spread $-q_a$ onto cavity a

$$\rightarrow \boxed{\sigma_a = -\frac{q_a}{4\pi a^2}}$$

(ii) to cancel q_b, spread $-q_b$ onto cavity b

$$\rightarrow \boxed{\sigma_b = -\frac{q_b}{4\pi b^2}}$$

(iii) for charge neutrality spread (q_a+q_b) onto sphere's surface. To maintain $E=0$ in the conductor spread it uniformly

$$\rightarrow \boxed{\sigma_R = \frac{q_a+q_b}{4\pi R^2}}$$

b) what is the field outside of the conductor? A spherical shell of charge (q_a+q_b)

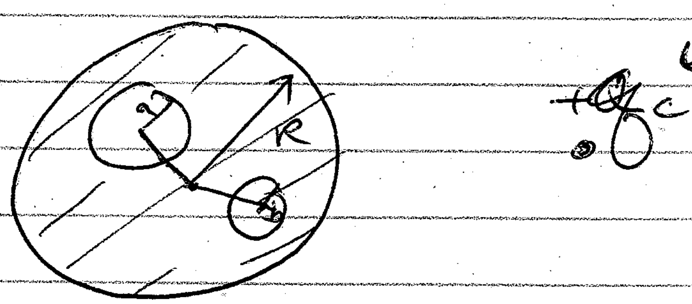
$$\rightarrow \boxed{\vec{E}_{r>R} = \frac{1}{4\pi\epsilon_0}\frac{q_a+q_b}{r^2}\hat{r}}$$

c) What is the field in each cavity?

(i) $\vec{E}_a = \frac{1}{4\pi\epsilon_0} \frac{q_a}{r_a^2} \hat{r}_a$, where $\vec{r}_a$ is the vector from the center of the cavity a

(ii) $\vec{E}_b = \frac{1}{4\pi\epsilon_0} \frac{q_b}{r_b^2} \hat{r}_b$, where $\vec{r}_b$ is the vector from the center of cavity b

d) Consider the addition of charge q_c



Q: What happens? A surface charge is induced at $r = R$ (sphere's surface) which cancels the field of q_c in the conductor.

(18) <u>Yukaw Pot^{l} (Problem 2.46 in text)</u>

$$V = K\frac{e^{-r/r_0}}{r}$$

$$\rightarrow \vec{E} = -\vec{\nabla}V = -K\hat{r}\left[-\frac{e^{-r/r_0}}{rr_0} - \frac{e^{-r/r_0}}{r^2}\right]$$

$$\boxed{\vec{E} = Ke^{-r/r_0}\left[\frac{\hat{r}}{rr_0} + \frac{\hat{r}}{r^2}\right]}$$

a) <u>find $\rho(r)$</u>

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\rightarrow \rho/\epsilon_0 = \left[K\frac{e^{-r/r_0}}{r_0}\left(\vec{\nabla}\cdot\frac{\hat{r}}{r}\right) + \frac{\hat{r}}{r}\cdot\vec{\nabla}K\frac{e^{-r/r_0}}{r_0}\right.$$

$$\left. + Ke^{-r/r_0}\left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2}\cdot\vec{\nabla}Ke^{-r/r_0}\right]$$

$$= K\frac{e^{-r/r_0}}{r_0}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2/r\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(K\frac{e^{-r/r_0}}{r_0}\right)$$

$$+ Ke^{-r/r_0}4\pi\delta^3(\vec{r}) + \frac{1}{r^2}\left(\frac{\partial}{\partial r}Ke^{-r/r_0}\right)$$

$$= \frac{Ke^{-r/r_0}}{r_0}\left(\frac{1}{r^2} - \frac{1}{rr_0} + 4\pi\delta^3(\vec{r})r_0 - \frac{1}{r^2}\right)$$

$$\boxed{\frac{\rho}{\epsilon_0} = K\frac{e^{-r/r_0}}{r_0}\left(r_0 4\pi\delta^3(\vec{r}) - \frac{1}{rr_0}\right)}$$

$$b)\quad Q = \int_0^\infty \rho \, d^3x$$

$$= \frac{k\epsilon_0}{r_0} \int_0^\infty \left(4\pi \delta^3(\vec{r}) r_0 - \frac{1}{r r_0} \right) e^{-r/r_0} \, d^3x$$

$$= \left[\frac{4\pi k r_0}{r_0} - \frac{k}{r_0} \int_0^\infty \frac{e^{-r/r_0}}{r r_0} 4\pi r^2 dr \right] \epsilon_0$$

$$= \frac{4\pi k \epsilon_0}{r_0} \left[r_0 - \frac{1}{r_0} \int_0^\infty \underbrace{r e^{-r/r_0} dr}_{\text{do by parts}} \right]$$

let: $U = r$, $dV = e^{-r/r_0} dr$

$dU = dr \quad V = -r_0 e^{-r/r_0}$

$$\rightarrow Q = \frac{4\pi}{r_0} k\epsilon_0 \left[r_0 - \frac{1}{r_0} \left\{ \cancel{-r r_0 e^{-\frac{r}{r_0}} \Big|_0^\infty}^{\,0} + r_0 \int_0^\infty e^{-\frac{r}{r_0}} dr \right\} \right]$$

$$= \frac{4\pi \epsilon_0}{r_0} k \left[r_0 - \frac{1}{r_0} \left\{ r_0 \times (-r_0) e^{-r/r_0} \Big|_0^\infty \right\} \right]$$

$$= \frac{4\pi \epsilon_0}{r_0} k \left[r_0 + r_0 (0 - 1) \right]$$

$$= 0 \; !$$

the total charge of the configuration which leads to $V = \frac{k e^{-r/r_0}}{r}$ is 0; the object is neutral

(15) Find Binding Energy of the Configuration (see Prob. 2.33)

We have a point charge surrounded by a cloud

$$\rho(r) = 4\pi\epsilon_0 K e^{-r/r_0}\delta^3(\vec{r}) - K\epsilon_0 \frac{e^{-r/r_0}}{r r_0^2}$$

$$\boxed{\rho(r) = 4\pi\epsilon_0 K\delta^3(\vec{r}) - K\epsilon_0 \frac{e^{-r/r_0}}{r r_0^2}}$$

drop e^{-r/r_0} because $\delta^3(\vec{r})$ is only nonzero when $r=0 \rightarrow e^{-r/r_0} = 1$

a) Let's consider the problem in 2 steps. Bring in charge onto point charge ignoring "cloud"; Bring in cloud ignoring point charge (why is this valid?).

(i) $Q_p = \int \rho_p d^3x = \int 4\pi\epsilon_0 K\delta^3(\vec{r})d^3\vec{x} = 4\pi\epsilon_0 K$

$$\Rightarrow \vec{E}_p = \frac{K\hat{r}}{r^2} \Rightarrow V_p = \frac{K}{r} \left(\text{from } dV_p = -\vec{E}_p \cdot d\vec{r}\right)$$

(ii) Build up cloud layer-by-layer. This involves we lay on shells of radius R onto the configuration; the charge laid on is, "cloud" density at R

$$dq = 4\pi R^2 dR\,\rho(R)$$

from prob. 2.33

$$\rightarrow W_{pc} = \int dq\, V(R,R)$$

$$= \int_0^\infty 4\pi R^2\left(-\epsilon_0 K\frac{e^{-R/r_0}}{R r_0^2}\right) dR\, \frac{K}{R}$$

$$= -\frac{4\pi\epsilon_0 K}{r_0^2}\int_0^\infty \cancel{R} e^{-R/r_0} dR\,\frac{K}{\cancel{R}}$$

$$W_{pe} = +\frac{4\pi\epsilon_0 K^2}{r_0^2}\left(+r_0 e^{-R/r_0}\Big|_0^\infty\right)$$

recall from a)

$$= -\frac{4\pi\epsilon_0 K^2}{r_0} \longleftarrow Q_p = 4\pi\epsilon_0 K \rightarrow K^2 = \left(\frac{Q_p}{4\pi\epsilon_0}\right)^2$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{Q_p^2}{r_0}$$

(ii) Begin cloud taking account of the cloud $V_{cl}(r,r)$;

$\Rightarrow dq = 4\pi R^2 \rho(R)\,dR$ — not total $V(r)$

is the same, but we now need $V_{cl}(R,R)$, cloud potential

Use

$$V_c = \frac{1}{4\pi\epsilon_0}\frac{\int_0^R \rho_c\, d^3x}{R}$$

this is true because in this calculation we need only consider the charge interior to the radius R. This is different than if we had tried to apply $W = \frac{1}{2}\int \rho V d^3x$. why?

$$= \frac{1}{4\pi\epsilon_0 R}\left(\frac{Q_p R}{r_0}e^{-\frac{R}{r_0}} + Q_p e^{-\frac{R}{r_0}} - Q_p\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\left(\frac{Q_p}{R}\right)\left(1 - e^{-\frac{R}{r_0}}\left[\frac{R}{r_0}+1\right]\right)$$

and,

$$W_{cc} = \int_0^\infty 4\pi R^2 \left(-K\epsilon_0 \frac{e^{-\frac{R}{r_0}}}{R r_0^2}\right) dR \times$$

$$\times \frac{1}{4\pi\epsilon_0}\left(-\frac{Q_p}{R}\right)\left(1 - \left[1+\frac{R}{r_0}\right] e^{-\frac{R}{r_0}}\right)$$

$$= \frac{KQ_p}{r_0^2}\int_0^\infty e^{-\frac{R}{r_0}} dR \left(1 - \left[1+\frac{R}{r_0}\right] e^{-\frac{R}{r_0}}\right)$$

$$\cancel{= \frac{KQ_p}{r_0^2}\left[-r_0 e^{-\frac{R}{r_0}}\Big|_0^\infty - \int_0^\infty \left(1 - \left[1+\frac{R}{r_0}\right] e^{-\frac{R}{r_0}}\right) dR\right.}$$

$$= \frac{KQ_p}{r_0^2}\left[-r_0 e^{-\frac{R}{r_0}}\Big|_0^\infty - \int_0^\infty \left(1+\frac{R}{r_0}\right) e^{-\frac{2R}{r_0}} dR\right]$$

$$= \frac{KQ_p}{r_0^2}\left[+r_0 + \frac{r_0}{2} e^{-\frac{2R}{r_0}}\Big|_0^\infty - \int_0^\infty \frac{R}{r_0} e^{-2R/r_0} dR\right]$$

$$= K\frac{Q_p}{r_0^2}\left(r_0 - \frac{r_0}{2}\right) - \frac{KQ_p}{r_0^2}\int_0^\infty \frac{R}{r_0} e^{-2R/r_0} dR$$

$$= \frac{KQ_p}{2r_0} - \frac{KQ_p}{r_0^3}\int_0^\infty R e^{-2R/r_0}$$

$$W_{cc} = \frac{kQ_p}{2r_0} - \frac{kQ_p}{r_0^3}\left[-\frac{r_0}{2} R e^{-\frac{2R}{r_0}} \Big|_0^\infty + \frac{r_0}{2}\int_0^\infty e^{-\frac{2R}{r_0}} dR \right]$$

$$= \frac{kQ_p}{2r_0} - \frac{kQ_p}{r_0^3}\left[0 - \frac{r_0^2}{4} e^{-\frac{2R}{r_0}} \Big|_0^\infty \right]$$

$$= \frac{kQ_p}{2r_0} + \frac{kQ_p}{4r_0}(0-1)$$

$$\boxed{W_{cc} = \frac{kQ_p}{4r_0}} > 0 \Rightarrow \text{electron in bond}$$

Binding energy is

$$W_{Total} = W_{pe} + W_{cc} \qquad 4\pi\epsilon_0 k = Q_p$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{Q_p^2}{r_0} + \frac{kQ_p}{4r_0}$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{Q_p^2}{r_0} + \frac{1}{4\pi\epsilon_0}\frac{Q_p^2}{4r_0}$$

$$\boxed{W_{Total} = -\frac{1}{4\pi\epsilon_0}\frac{3Q_p^2}{4r_0}}$$