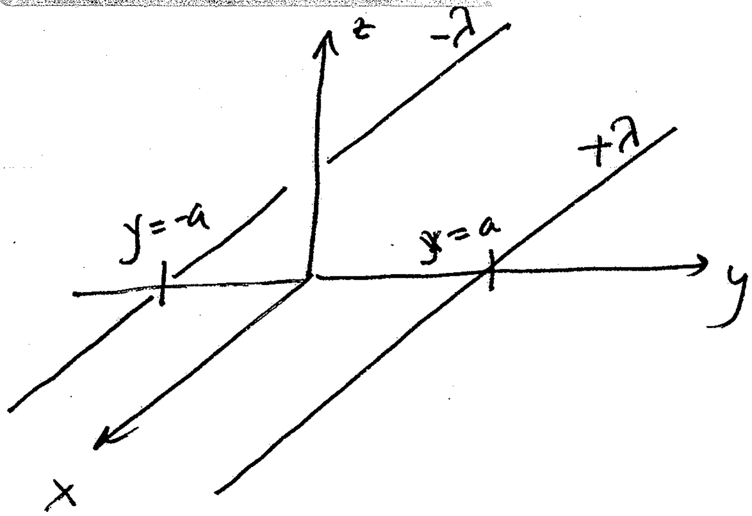
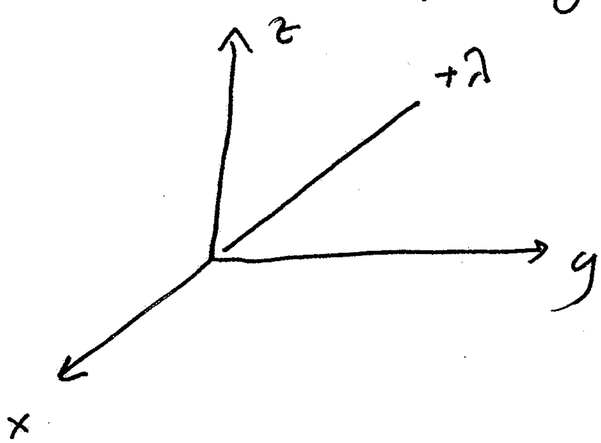
# Problem 2.47




a) Find $V(x,y,z)$ using the origin as your reference

Consider 1 line charge along x-axis




By Gauss's law,

$$\vec{E}_s = \hat{s}\,\frac{\lambda}{2\pi\epsilon_0 s}$$

and $dV = -\vec{E}_s \cdot d\vec{s}$

$$V = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s}{s_0}\right)$$

$s_0$ ← reference pt.

Translate V to $y = \pm a$ (Use Cartesian coordinates)

(i) $V(x,y,z) = -\dfrac{\lambda}{2\pi\epsilon_0}\ln\left[\dfrac{\sqrt{y^2+z^2}}{s_0}\right]$ is "x-axis sol$^{\underline{n}}$"

Translate to $y = \pm a$ and sum for 2 line charges,

$$V(x,y,z) = -\frac{\lambda}{2\pi\epsilon_0}\left\{\ln\left[\frac{\sqrt{(y-a)^2+z^2}}{s_0}\right] - \ln\left[\frac{\sqrt{(y+a)^2+z^2}}{s_0}\right]\right.$$

2$^{\underline{nd}}$ line has $\lambda' = -\lambda$

$$\boxed{V(x,y,z) = -\frac{\lambda}{2\pi\epsilon_0} \ln\left[\frac{\sqrt{(y-a)^2+z^2}}{\sqrt{(y+a)^2+z^2}}\right]}$$

b) Show that the equipotentials are circular cylinders and locate the axis and radius of the cylinder corresponding to a given potential $V_0$.

$V(x,y,z) = V_0 \Rightarrow$ defines equipotential surfaces.

$$\Rightarrow -\frac{2\pi\epsilon_0 V_0}{\lambda} = \ln\left[\frac{\sqrt{(y-a)^2+z^2}}{\sqrt{(y+a)^2+z^2}}\right]$$

$$-\frac{4\pi\epsilon_0 V_0}{\lambda} = \ln\left[\frac{(y-a)^2+z^2}{(y+a)^2+z^2}\right] \leftarrow \text{took square root outside } \underline{\ln}.$$

$$\underset{\text{constant}}{C_0} = \exp\left(-\frac{4\pi\epsilon_0 V_0}{\lambda}\right) = \left(\frac{(y-a)^2+z^2}{(y+a)^2+z^2}\right)$$

$$\Rightarrow C_0\left[(y+a)^2+z^2\right] = (y-a)^2+z^2$$

$$z^2(C_0-1) + \left\{y^2(C_0-1) + y\left(2aC_0+2a\right) + a^2(C_0-1)\right\} = 0$$

$$z^2 + \underbrace{\left\{y^2 + y\,2a\left(\frac{C_0+1}{C_0-1}\right) + a^2\right\}}_{\text{complete the square}} = 0$$

$$y^2 + 2a\left(\frac{c_0+1}{c_0-1}\right)y + a^2 = 0$$

$$\left(y + a\left[\frac{c_0+1}{c_0-1}\right]\right)^2 + \left(a^2 - a^2\left[\frac{c_0+1}{c_0-1}\right]^2\right) = 0$$

$$\left(y - a\left[\frac{1+c_0}{1-c_0}\right]\right)^2 + a^2\left(\frac{c_0^2-2c_0+1-c_0^2-2c_0-1}{(c_0-1)^2}\right) = 0$$

$$\left(y - a\left[\frac{1+c_0}{1-c_0}\right]\right)^2 + a^2\left(\frac{-4c_0}{(c_0-1)^2}\right) = 0$$

So, we have

$$\boxed{\left(y - a\left[\frac{1+c_0}{1-c_0}\right]\right)^2 + z^2 = \frac{4a^2c_0}{(c_0-1)^2}}$$

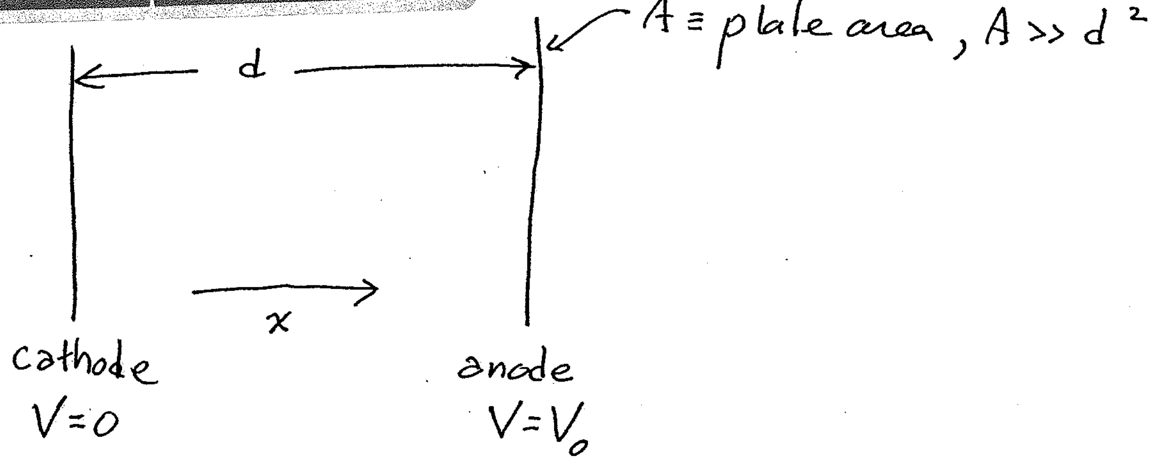
the above is an equation for a circle centered at

$$y = a\left(\frac{1+c_0}{1-c_0}\right) \quad \text{and} \quad z = 0$$

with radius

$$R = \frac{2a}{|c_0-1|}\sqrt{c_0}$$

# Problem 2.48




a) Poisson's Equation:

$$\boxed{\frac{\partial^2}{\partial x^2} V = -\rho(x)/\epsilon_0}$$

b) $E = \frac{1}{2} m v^2 + q V(x)$ : Energy equation for $e^-$'s

at $x=0$, $V=0$, $v^2=0$ $\rightarrow$ $E=0$

$q=-e$

$$\Longrightarrow \frac{1}{2} m v^2 = -q V(x) \rightarrow v^2(x) = -\frac{2q}{m} V(x)$$

$$\boxed{v^2(x) = \frac{2e}{m} V(x)}$$

c) In steady state, $I_0$ is independent of $x$. What is the relation between $\rho$ and $v(x)$?

$$J_0 = \frac{I_0}{A} = \rho v \Rightarrow \boxed{I_0 = \rho v A = \text{constant} < 0}$$

$e^-$'s move to right

d) Using a, b, c find a differential equation for $V$.

$$\frac{\partial^2}{\partial x^2} V(x) = -\frac{1}{\epsilon_0}\left[\frac{I_0}{\rho v A}\right], \text{ substituting for } \rho$$

$$= -\frac{I_0}{\epsilon_0 A}\left[\frac{m}{2eV(x)}\right]^{1/2}, \text{ substituting for } v(x) \text{ velocity}$$

$$\boxed{\frac{\partial^2}{\partial x^2} V(x) + \sqrt{\frac{m I_0^2}{\epsilon_0^2 A^2 2e}}\, V^{-1/2}(x) = 0}$$

e) find $V(x)$ as a function of $x$, $V_0$, and $d$

Guess that

$$V(x) = a x^b$$

is a solution to the differential equation

$$a b(b-1) x^{b-2} + \sqrt{\frac{m I_0^2}{\epsilon_0^2 A^2 2e}}\, \frac{1}{a^{1/2}} x^{-b/2} = 0$$

$$a b(b-1) x^{\frac{3b}{2}-2} + \sqrt{\frac{m I_0^2}{2\epsilon_0^2 A^2 e}}\, \frac{1}{\sqrt{a}} = 0$$

$$\Longrightarrow b = 4/3$$

and so,

$$\frac{4}{3} a \left(\frac{1}{3}\right) + \frac{1}{\sqrt{a}} \sqrt{\frac{m I_0^2}{2\epsilon_0^2 A^2 e}} = 0$$

$$\Rightarrow a^{3/2} = -\frac{9}{4} \frac{I_0}{\epsilon_0 A} \sqrt{\frac{m}{2e}}$$

the potential is

$$V(x) = \left[\frac{9}{4} \frac{I_0}{\epsilon_0 A} \sqrt{\frac{m}{2e}}\right]^{\frac{2}{3}} x^{4/3}$$

Hmm, we want $V(x)$ in terms of $x$, $V_0$, and $d$. Let

$$V(x) = +K x^{4/3}, \text{ where } K = \left[\frac{9 I_0}{4\epsilon_0 A} \sqrt{\frac{m}{2e}}\right]^{2/3}$$

at $x=d$, $V=V_o$

$$V_o = +Kd^{4/3} \longrightarrow +K = V_o/d^{4/3}$$

and

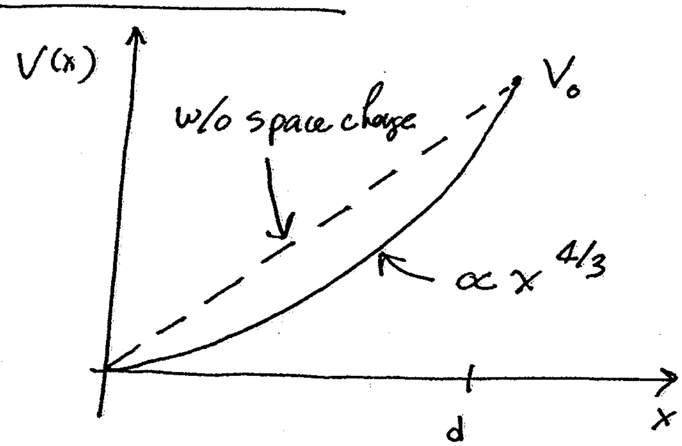
$$\boxed{V(x) = \left(\frac{V_o}{d^{4/3}}\right) x^{4/3}}$$

find $\rho(x)$ and $v(x)$

$$v^2(x) = \frac{2e}{m}\left(\frac{V_o}{d^{4/3}}\right) x^{4/3}$$

$$\rho(x) = \left(\frac{I_o}{A}\right)\left[\frac{2e}{m}\frac{V_o}{d^{4/3}}\right]^{-\frac{1}{2}} x^{-2/3}$$

Plot $V(x)$




f) Show that $I_o = K V_o^{3/2}$ and find $K$.

$$V(x) = \left[\frac{9I_o}{4\epsilon_0 A}\sqrt{\frac{m}{2e}}\right]^{2/3} x^{4/3}$$

<u>at $x = d$, $V = V_0$</u>

$$V_0 = \left[\frac{9 I_0}{4\epsilon_0 A}\sqrt{\frac{m}{2e}}\right]^{2/3} d^{4/3}$$

$$\Rightarrow \underbrace{I_0}_{>0} = \underbrace{d^{-2}\left(\frac{4\epsilon_0 A}{9}\sqrt{\frac{2e}{m}}\right)}_{K} V_0^{3/2}$$

(we defined $I_0$ in [illegible])

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$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\hat{r}$$

(a) What is the electric field of a charge distribution $\rho$?

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\, d^3x}{\mathscr{r}^2}\left(1+\frac{\mathscr{r}}{\lambda}\right)e^{-\frac{\mathscr{r}}{\lambda}}\hat{\mathscr{r}}}$$

(b) Does this electric field admit a scalar potential?

$\boxed{\text{Yes,}}$ because

$$\vec{\nabla}\times\vec{E} = \frac{1}{4\pi\epsilon_0}\vec{\nabla}\times\underbrace{\int\frac{\rho\, d^3x}{\mathscr{r}^2}\left(1+\frac{\mathscr{r}}{\lambda}\right)e^{-\mathscr{r}/\lambda}\hat{\mathscr{r}}}_{}$$

radial function because

$$\hat{\mathscr{r}} = \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\ ;\ \vec{r}\,' \text{ integration variable}$$

and so, $\vec{\nabla}\times f(\mathscr{r})\hat{\mathscr{r}} = 0$

(c) $$-dV = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\hat{r}\cdot d\vec{r}$$

$$-V(r)+V_\infty = \frac{q}{4\pi\epsilon_0}\int_\infty^r\frac{(1+r/\lambda)}{r^2}e^{-r/\lambda}dr$$

$$= \cancel{\frac{q}{4\pi\epsilon_0}\left[-\frac{q}{r}e^{-r/\lambda} + \frac{1}{\lambda}\left(\log r - \frac{r}{\lambda}+\frac{r^2}{4\lambda^2}-\frac{r^3}{18\lambda^3}+\dots\right)_r\right]}$$

$$= \frac{q}{4\pi\epsilon_0}\left[-\frac{e^{-r/\lambda}}{r} - \frac{1}{\lambda}\int_\infty^r\frac{e^{-r/\lambda}}{r}dr + \int_\infty \frac{e^{-r/\lambda}}{\lambda r}dr\right]$$

$$\boxed{V(r) = \underbrace{V_\infty}_{\to 0} + \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r} = \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r}}$$

d) $$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left[\frac{q}{4\pi\epsilon_0}\,\frac{1+r/\lambda}{r^2}\,e^{-r/\lambda}\,\hat{r}\right], \quad r\neq 0$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\cancel{r^2}\,\frac{q}{4\pi\epsilon_0}\left(\frac{1+r/\lambda}{\cancel{r^2}}\right)e^{-r/\lambda}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{1}{r^2}\left[-\frac{1}{\lambda}e^{-r/\lambda}\left(\cancel{1}+\frac{r}{\lambda}\right)+\cancel{\frac{1}{\lambda}e^{-r/\lambda}}\right]$$

$$= -\frac{q}{4\pi\epsilon_0}\left[+\frac{r}{\lambda^2 r^2}\,e^{-r/\lambda}\right]$$

$$= -\frac{q}{4\pi\epsilon_0\lambda^2}\left[\frac{e^{-r/\lambda}}{r}\right]$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = -\frac{1}{\lambda^2}V(r)} \quad \text{if } \vec{r}\neq 0$$

Suppose $r \to 0$

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left[\frac{q}{4\pi\epsilon_0}\left(\frac{1+r/\lambda}{r^2}\right)\left(1-\frac{r}{\lambda}\right)\hat{r}\right]$$

$$= \vec{\nabla}\cdot\left[\frac{q}{4\pi\epsilon_0}\left(\frac{\hat{r}}{r^2}-\cancel{\frac{1}{\lambda^2}\hat{r}}\right)\right]$$ 2nd order in $\left(\frac{r}{\lambda}\right)$; don't keep

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{q}{\epsilon_0}\delta^3(\vec{r})} - \cancel{\frac{q}{4\pi\epsilon_0\lambda^2}}\Big($$

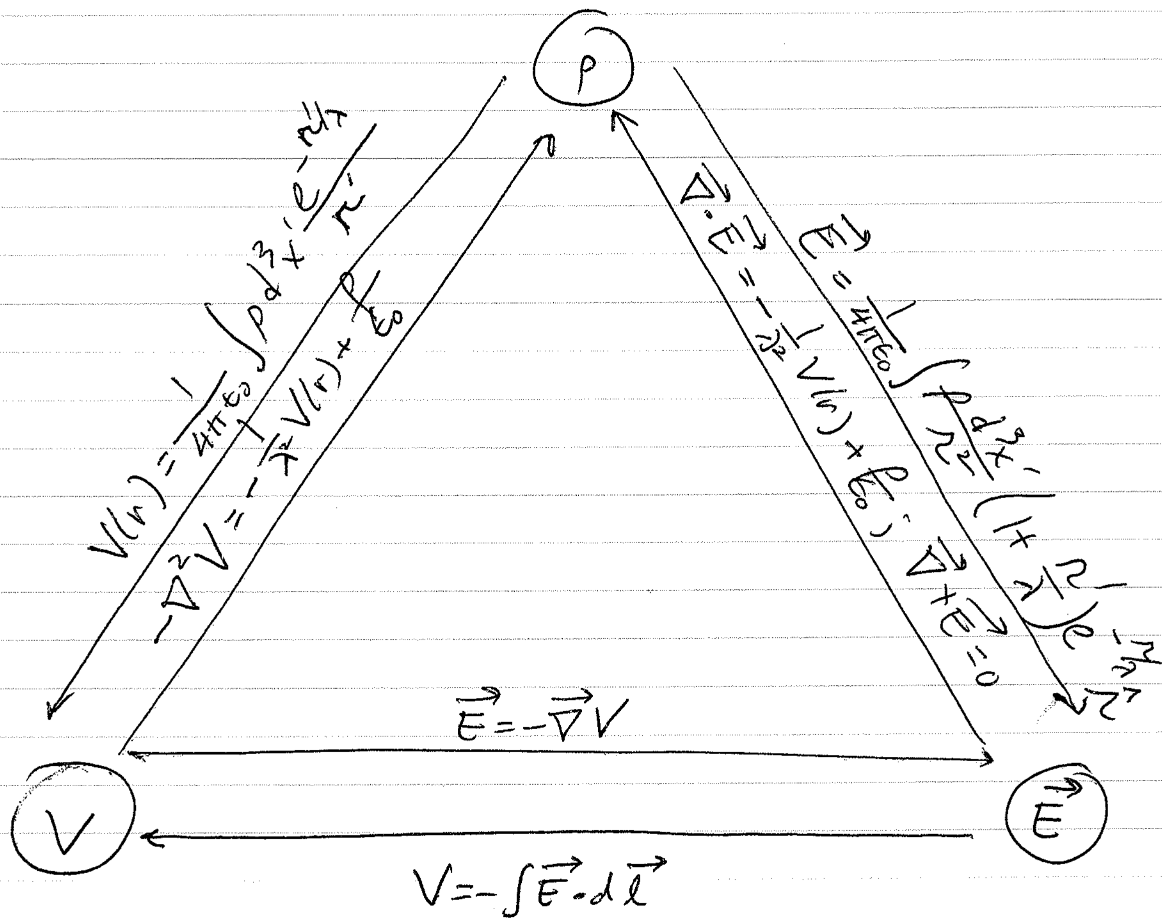
$$\Rightarrow \int(\vec{\nabla}\cdot\vec{E})\,d\tau = \int -\frac{1}{\lambda^2}V(r)\,d\tau + \int\frac{q}{\epsilon_0}\delta^3(\vec{r})\,d\tau$$

$$\boxed{\oint\vec{E}\cdot d\vec{S} = -\frac{1}{\lambda^2}\int V(r)\,d\tau + \frac{q}{\epsilon_0}}$$

(e) Show that the answer to part (d) generalizes to any volume V. The result in (d) directly generalizes because the δ-function gives the charge integral a contribution at only the locations of the charge. Then we have

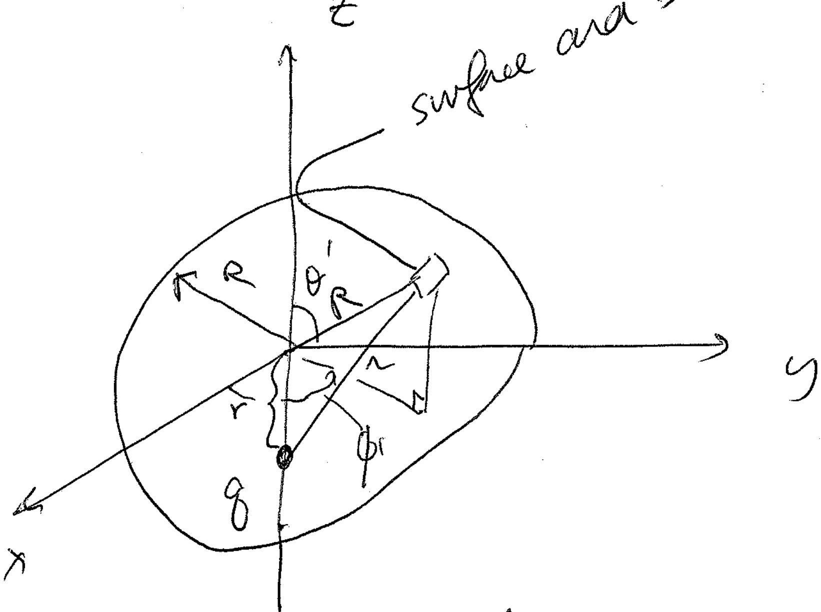
$$\boxed{\oint \vec{E}\cdot d\vec{S} + \frac{1}{\lambda^2}\int V\,d\tau = \frac{Q_{enc}}{\epsilon_0}}$$

(f) Triangle Diagram



# Problem 3.1

surface area $= R^2 \sin\theta' d\theta' d\phi'$




find the average V on the shell of radius R of the point charge q located at $z = -r$

$$\langle V \rangle = \frac{\frac{1}{4\pi\epsilon_0}\int \frac{q}{r}\left(R^2 \sin\theta' d\theta' d\phi'\right)}{4\pi R^2}$$

$$= \frac{q}{16\pi^2\epsilon_0}\int \frac{\sin\theta' d\theta' d\phi'}{\sqrt{r^2+R^2-2rR\cos(\pi-\theta')}}$$

$$= \frac{q}{16\pi^2\epsilon_0}2\pi\int_{1}^{-1}\frac{d(-\cos\theta')}{\sqrt{r^2+R^2+2rR\cos\theta'}}$$

$$= \frac{-q}{8\pi\epsilon_0}\left[2\frac{1}{2Rr}\sqrt{r^2+R^2+2Rr\cos\theta'}\right]_1^{-1}$$

$$= -\frac{q}{8\pi\epsilon_0 Rr}\left[\sqrt{(r-R)^2}-\sqrt{(r+R)^2}\right]$$

note: $r < R$

$$\langle V \rangle = -\frac{q}{8\pi\epsilon_0 Rr}\left[(R-r)-(R+r)\right] = \frac{q}{4\pi\epsilon_0 R}$$ ✓

Problem 3.3

(a) $\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}V\right) = \int 0\,ds$

$$\int \frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}V\right)ds = C_0$$

$$s\frac{\partial}{\partial s}V = C_0$$

$$\boxed{V = C_0 \ln s + C_1}$$

(b) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}V\right) = \int 0\,ds$

$$\int \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}V\right)ds = \int 0\,ds$$

$$r^2\frac{\partial}{\partial r}V = C_0$$

$$\boxed{V = \frac{-C_0}{r} + C_1}$$