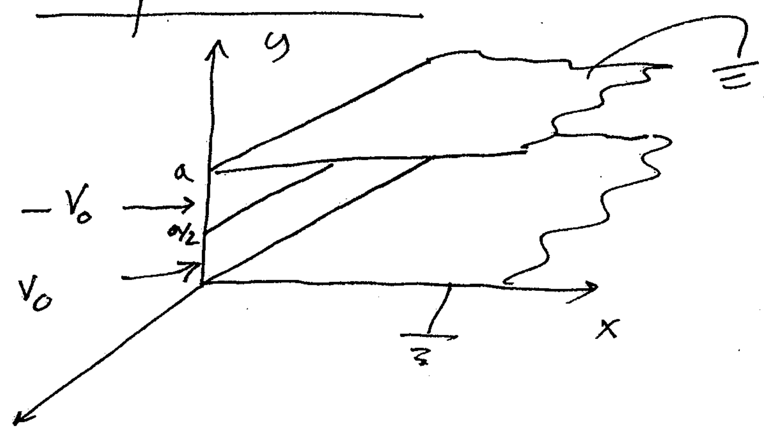
Prob 3.12

Find V in the infinite slot if the boundary at $x=0$ consists of 2 strips: one from $y=0$ to $y=a/2$, is held at constant potential V_0, and the other, from $y=a/2$ to a, is at potential $-V_0$



Use sol^n from ex 3.3 (up to imposition of $x=0$ BC)

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a}y\right) \qquad (3.30)$$

at $x=0$

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}y\right) = V_0(x=0,y)$$

multiply by $\sin\left(\frac{n'\pi}{a}y\right)dy$ and integrate over $y=[0,a]$

$$\int_0^a \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right)dy = \int_0^{\frac{a}{2}} V_0 \sin\left(\frac{n'\pi}{a}y\right)dy - \int_{\frac{a}{2}}^{a} V_0 \sin\left(\frac{n'\pi}{a}y\right)dy$$

$$\frac{a}{2}C_{n'} = V_0\frac{a}{n'\pi}\left[-\cos\left(\frac{n'\pi}{a}y\right)\Big|_0^{a/2} + \cos\left(\frac{n'\pi}{a}y\right)\Big|_{a/2}^{a}\right]$$

$$= \frac{aV_0}{n'\pi}\left[\left(-\cos\left(\frac{n'\pi}{2}\right)+1\right) + \left(\cos(n'\pi) - \cos\left(\frac{n'\pi}{2}\right)\right)\right]$$

$$= \frac{aV_0}{n'\pi}\left[1 + \cos(n'\pi) - 2\cos\left(\frac{n'\pi}{2}\right)\right]$$

$$\frac{a}{2} C_{n'} = \frac{aV_0}{n'\pi}\left[\left(\cos(n'\pi)+1\right) - 2\cos\left(\frac{n'\pi}{2}\right)\right]$$

$$= \frac{aV_0}{n'\pi} \begin{cases} 0 & n' = \text{odd} ;\ 4, 8, 12, 16, \cdots \\ 4 & n' = \quad 2, 6, 10, \cdots \end{cases}$$

$$= (4n+2), \ n = 0, 1, 2, 3, \cdots$$

$$\Longrightarrow C_{n'} = \frac{8V_0}{n'\pi}, \quad n' = (4n+2) \text{ where } n = 0, 1, 2, 3, \cdots$$

or

$$C_n = \frac{8V_0}{(4n+2)\pi}, \quad n = 0, 1, 2, \cdots$$

$$\boxed{V(x,y) = \sum_{n=0}^{\infty} \frac{8V_0}{(4n+2)\pi} e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a}y\right)}$$

Prob 3.13

For the infinite slot in Ex 3.3, determine $\sigma(y)$ at $x=0$ assuming it has a constant potential V_0

Use the "summed" sol^{n} for Ex 3.3

$$V(x,y) = \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right)$$

to find $\vec{E}(x=0,y)$ and $\therefore$ $\sigma(y) = \epsilon_0 E_x$

$$E_x = -\frac{\partial V(x,y)}{\partial x} = \frac{-2V_0}{\pi}\left[\frac{1}{1+\left\{\frac{\sin(\pi y/a)}{\sinh(\frac{\pi x}{a})}\right\}^2}\right] \times \left[\frac{-\sin(\frac{\pi y}{a})}{\sinh^2(\frac{\pi x}{a})}\,\frac{\pi}{a}\cosh\frac{\pi x}{a}\right]$$

at $x=0$

$$E_x = \frac{2V_0}{a}\sin\left(\frac{\pi y}{a}\right)\frac{1}{1+\sin^2(\frac{\pi y}{a})}$$

and

$$\boxed{\sigma(y) = \frac{2V_0\epsilon_0}{a}\left[\frac{\sin(\frac{\pi y}{a})}{1+\sin^2(\frac{\pi y}{a})}\right]}$$

Prob 3.16

Derive $P_3(\mu)$ from the Rodrigues formula & check that $P_3(\mu)$ satisfies the angular portion of the Laplace equation. Verify that $P_3(\mu)$ and $P_1(\mu)$ are orthogonal.

a) $$P_\ell(\mu) = \frac{1}{2^\ell \ell!}\left(\frac{d}{d\mu}\right)^\ell (\mu^2-1)^\ell$$

$$P_3(\mu) = \frac{1}{2^3 3!}\frac{d^3}{d\mu^3}(\mu^2-1)^3$$

$$= \frac{1}{8\times 6}\frac{d^2}{d\mu^2}\left(3\left[\mu^2-1\right]^2 2\mu\right)$$

$$= \frac{3}{48}\frac{d}{d\mu}\left[2(\mu^2-1)^2 + 4\mu(\mu^2-1)2\mu\right]$$

$$= \frac{1}{8}\left[2(\mu^2-1)2\mu + 8\mu(\mu^2-1) + 4\mu^2(2\mu)\right]$$

$$= \left[\frac{1}{2}(\mu^2-1)\mu + (\mu^2-1)\mu + (\mu^3)\right]$$

$$= \left[\frac{5}{2}\mu^3 - \frac{3}{2}\mu\right]$$

$$\boxed{P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu)}$$

b) $$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin\theta\,\Theta = 0 \qquad (3.60)$$

$$\frac{d}{d\theta}\left[\left(\sin\theta\frac{d}{d\theta}\right)\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right)\right] + 12\sin\theta\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) = 0$$

$$\frac{d}{d\theta}\left(\sin\theta\left\{-\frac{15}{2}\cos^2\theta\sin\theta + \frac{3}{2}\sin\theta\right\}\right) + 30\sin\theta\cos^3\theta - 18\sin\theta\cos\theta = 0$$

$$\frac{d}{d\theta}\left(-\frac{15}{2}\cos^2\theta\sin^2\theta + \frac{3}{2}\sin^2\theta\right) + 30\sin\theta\cos^3\theta - 18\sin\theta\cos\theta = 0$$

$$+15\cos\theta\sin^3\theta - 15\cos^3\theta\sin\theta + 3\sin\theta\cos\theta + 30\sin\theta\cos^3\theta - 18\sin\theta\cos\theta = 0$$

$$\sin\theta\cos\theta\left(15\sin^2\theta - \cos^2\theta + 3 + 30\cos^2\theta - 18\right) = 0$$

$$\sin\theta\cos\theta\left[\underbrace{15-15\cos^2\theta-15\cos^2\theta+3+30\cos^2\theta-18}_{\to 0\text{, as required}}\right] = 0$$

c) $\int_{-1}^{1} P_1(\mu)P_3(\mu)\,d\mu$

$= \int_{-1}^{1} \mu\,\frac{1}{2}(5\mu^3-3\mu)\,d\mu$

$= \frac{1}{2}\int_{-1}^{1}(5\mu^4-3\mu^2)\,d\mu$

$= \frac{1}{2}\left(\frac{5}{5}\mu^5-\frac{3}{3}\mu^3\right)\Big|_{-1}^{1}$

$= \frac{1}{2}\left(1-(-1)-1+(-1)\right)$

$= 0$

Prob 3.18

Find the potential everywhere for sphere of radius R whose surface potential is given by

$$V_0 = k\cos\theta$$

Find $\sigma(\theta)$ as well as $V(r,\theta)$.

Sol^n

(i) Sol^n, $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\theta)$

(ii) Boundary Conditions

a) $V \to 0$ as $r \to \infty$

b) V is regular at $r=0$

c) $V(r=R,\theta) = k\cos 3\theta$

① Outer Sol^n $(r>R)$, $A_l \to 0$

$$V^{>}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l^{>}}{r^{l+1}} P_l(\theta)$$

② Inner Sol^n $(r<R)$, $B_l \to 0$

$$V^{<}(r,\theta) = \sum_{l=0}^{\infty} A_l^{<} r^l P_l(\theta)$$

③ Surface, $r=R$

replace w/ P_l's

$$V(r=R,\theta) = k\cos 3\theta = (4\cos^3\theta - 3\cos\theta)k$$

$$= k\left(4\left[\frac{1}{5}(2P_3(\theta) + 3\cos\theta)\right] - 3\cos\theta\right)$$

$$= k\left(\frac{8}{5}P_3(\theta) - \frac{3}{5}P_1(\theta)\right)$$

$\underline{r<R}$

$$\Rightarrow \left(8P_3(\theta) - 3P_1(\theta)\right)\frac{k}{5} = \sum_{l=0}^{\infty} A_l^< P_l(\theta) R^l \quad , r<R$$

$$= A_0^< + A_1^< R P_1(\theta) + A_2^< R^2 P_2(\theta) + A_3^< R^3 P_3(\theta) + \cdots$$

match coefficients of like P_l's.

$$\Rightarrow A_0^< = A_2^< = 0 \quad \text{and} \quad A_l^< = 0 \text{ if } l > 3$$

and

$$A_1^< = -\frac{3k}{5R} \quad , \quad A_3^< = \frac{8k}{5R^3}$$

and
$$\boxed{V^<(r,\theta) = -\frac{3k}{5R} r\mu + \frac{8k}{5R^3} r^3 P_3(\theta)}$$

$\underline{r>R}$

$$\Rightarrow k\left(\frac{8}{5}P_3(\theta) - \frac{3}{5}P_1(\theta)\right) = \sum_{l=0}^{\infty} \frac{B_l^>}{R^{l+1}} P_l(\theta)$$

match coefficients of P_l's

$$\Rightarrow B_1^> = -\frac{3k}{5}R^2 \quad , \quad B_3^> = \frac{8k}{5}R^4 \quad , \quad B_l^> = 0 \text{ otherwise}$$

$$V^>(r,\theta) = -\frac{3k}{5}\frac{R^2}{r^2}\mu + \frac{8k}{5}\frac{R^4}{r^4}P_3(\theta)$$

Find $\sigma(\theta)$

$$\Delta E_r = \frac{\sigma}{\epsilon_0}$$

$$\Rightarrow \frac{\sigma}{\epsilon_0} = -\frac{\partial V^>}{\partial r}\bigg|_{r=R} - \left(-\frac{\partial V^<}{\partial r}\right)\bigg|_{r=R}$$

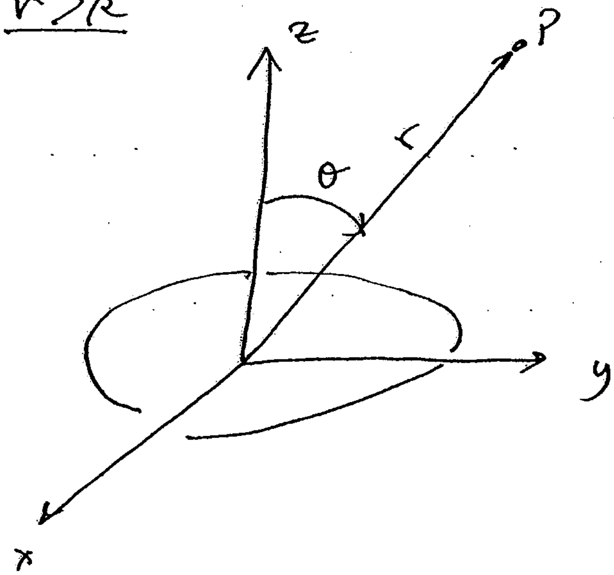
$$= -\left[+\frac{6k}{5R}\mu - \frac{32k}{5R}P_3(\mu)\right] - \left[+\frac{3k}{5R}\mu - \frac{24k}{5R}P_3(\mu)\right]$$

$$= -\frac{9k}{5R}\mu + \frac{56k}{5R}P_3(\mu)$$

$$\sigma = +\epsilon_0\left[-\frac{9k}{5R}\mu + \frac{28k}{5R}(5\mu^3 - 3\mu)\right]$$

Prob 3.21

a) Find the $V(r,\theta)$ for a charged disk off axis. Assume $r > R$



Use:

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\theta)$$

and

$$V(r,\theta) = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right] \quad \theta = 0$$

to find the lowest order terms.

Solution:

(i) note: $P_\ell(\mu = 1) = 1$, where $\mu = \cos\theta$

$$\Rightarrow V(r,\theta) = V(r,\theta=0) = V(r,\mu=1) \text{ and so,}$$

$$V(r,0) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\theta=0) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}}$$

(ii) we also have

$$V(r,\theta=0) = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right]$$

$$= \frac{\sigma}{2\epsilon_0}\left[r\sqrt{1+\left(\frac{R}{r}\right)^2} - r\right]$$

$$\approx \frac{\sigma}{2\epsilon_0}\left[r\left(1 + \frac{1}{2}\frac{R^2}{r^2} - \frac{1}{8}\frac{R^4}{r^4} + \frac{1}{16}\frac{R^6}{r^6}\right) - r\right]$$

$$= \frac{\sigma}{2\epsilon_0}\left[\frac{1}{2}\frac{R^2}{r} - \frac{1}{8}\frac{R^4}{r^3} + \frac{1}{16}\frac{R^6}{r^5}\right]$$

$$= \frac{\sigma R^2}{4r\epsilon_0}\left[1 - \frac{1}{4}\frac{R^2}{r^2} + \frac{1}{8}\frac{R^4}{r^4}\right]$$

$$\Rightarrow \quad B_0 = \frac{\sigma R^2}{4\epsilon_0}$$

$$B_2 = -\frac{\sigma R^4}{16\epsilon_0}$$

$$B_4 = \frac{\sigma R^6}{32\epsilon_0}$$

$$\vdots$$

ad $B_\ell = 0$, ℓ odd

so that

$$V(r,\theta) \approx \sum_{\ell \text{ even}}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\theta)$$

b) <u>Find $V(r,\theta)$ for $r < R$</u>

(i) given, $V(r,\theta=0) = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right]$

$$\approx \frac{\sigma R}{2\epsilon_0}\left[1 + \frac{1}{2}\frac{r^2}{R^2} - \frac{1}{8}\frac{r^4}{R^4} + \frac{1}{16}\frac{r^6}{R^6} + \cdots - \frac{r}{R}\right]$$

(ii) $V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\theta)$

<u>let $\theta = 0$</u> $\Rightarrow P_\ell(1) = 1$

above xy plane

$$\left\{ \begin{aligned} A_0^> &= \frac{\sigma R}{2\epsilon_0} \\ A_1^> &= -\frac{\sigma}{2\epsilon_0} \\ A_2^> &= \frac{\sigma R}{4\epsilon_0 R^2} = \frac{\sigma}{4\epsilon_0 R} \\ A_3^> &= 0 \end{aligned} \right.$$

$$A_4^> = -\frac{\sigma}{16\epsilon_0 R^3}$$

$$\vdots$$

$$\text{let } \theta = \pi \rightarrow P_\ell(-1) = (-1)^\ell$$

below xy plane

$$\left\{ \begin{aligned} A_0^< &= \frac{\sigma R}{2\epsilon_0} \\ A_1^< &= \frac{\sigma}{2\epsilon_0} \\ A_2^< &= \frac{\sigma}{4\epsilon_0 R} \\ A_3^< &= 0 \end{aligned} \right.$$

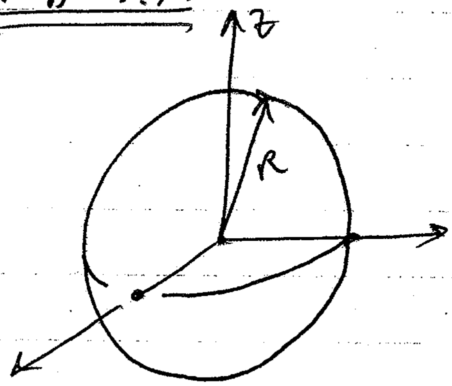
$$A_4^< = -\frac{\sigma}{16\epsilon_0 R^3}$$

$$\vdots$$

Sol^{ns} are

$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty} \begin{pmatrix} A_\ell^> \\ A_\ell^< \end{pmatrix} r^\ell P_\ell(\theta)}$$

Prob 3.22



$$\sigma = \begin{cases} \sigma_0 & 1 \geq \mu \geq 0 \\ -\sigma_0 & 0 \geq \mu \geq -1 \end{cases}$$

① Find Φ inside and outside of the sphere.

General Solution: $\Phi(r,\mu) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\mu)$

(i) Outside, as $r \to \infty$, $\Phi \to 0 \Rightarrow A_\ell = 0$

(ii) Inside, as $r \to 0$, Φ remains finite $\Rightarrow B_\ell = 0$

$$\Phi = \begin{cases} \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\mu) & r > R \\ \sum_\ell A_\ell r^\ell P_\ell(\mu) & r < R \end{cases}$$

(iii) at $r = R$, we have σ (charged sheet)

$\Rightarrow \Delta E_r = \frac{\sigma}{\epsilon_0}$ and $\Phi(r>R) = \Phi(r<R)$

(A) Continuity of Φ

$$\Rightarrow \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\mu) = \sum_\ell A_\ell r^\ell P_\ell(\mu) \quad \text{at } r=R$$

and term-by-term cancellation is required (as can also be shown explicitly)

$$A_\ell = \frac{B_\ell}{R^{2\ell+1}}$$

$$\Rightarrow \Phi = \begin{cases} \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\mu) & r>R \\ \sum_\ell \frac{B_\ell}{R^{2\ell+1}} r^\ell P_\ell(\mu) & r<R \end{cases}$$

(B) Jump in the "normal" component of $\vec{E}$.

$$-\frac{\partial \Phi_{out}}{\partial r}\Big|_{r=R} - \left(-\frac{\partial \Phi_{in}}{\partial r}\right)\Big|_{r=R} = \frac{\sigma}{\epsilon_0}$$

$$\sum_\ell (\ell+1)\frac{B_\ell}{R^{\ell+2}} P_\ell(\mu) + \sum_\ell \ell \frac{B_\ell R^{\ell-1}}{R^{2\ell+1}} P_\ell(\mu) = \frac{\sigma}{\epsilon_0}$$

$$\sum_\ell P_\ell(\mu)\left[\frac{B_\ell}{R^{\ell+2}}(\ell+1) + \frac{B_\ell}{R^{\ell+2}}\ell\right] = \frac{\sigma}{\epsilon_0}$$

$$\sum_\ell P_\ell(\mu)\left[\frac{B_\ell}{R^{\ell+2}}(2\ell+1)\right] = \frac{\sigma}{\epsilon_0}$$

multiply through by $P_m(\mu)d\mu$ and integrate over $[-1,1]$

$$\sum_l \int_{-1}^{1} (2l+1)\frac{B_l}{R^{l+2}} P_l(\mu)P_m(\mu)\,d\mu = \int_{-1}^{0} -\frac{\sigma_0}{\epsilon_0} P_m(\mu)\,d\mu + \int_{0}^{1} \frac{\sigma_0}{\epsilon_0} P_m(\mu)\,d\mu$$

$$(2l+1)\frac{B_l}{R^{l+2}}\frac{2}{2l+1}\delta_{ml} = \frac{\sigma_0}{\epsilon_0}\left[-\int_{-1}^{0} P_m(\mu)\,d\mu + \int_{0}^{1} P_m(\mu)\,d\mu\right]$$

Consider:

m	$P_m(\mu)$
0	1
1	μ
2	$\frac{1}{2}(3\mu^2-1)$
3	$\frac{1}{2}(5\mu^3-3\mu)$
4	$\frac{1}{8}(35\mu^4-30\mu^2+3)$
5	$\frac{1}{8}(63\mu^5-70\mu^3+15\mu)$
6	

Now find the above integrals:

$m=0$ $\quad \frac{\sigma}{\epsilon_0}\left[-\int_{-1}^{0} dy + \int_0^1 dy\right] = 0$

$m=1$ $\quad \frac{\sigma}{\epsilon_0}\left[-\int_{-1}^{0} y\,dy + \int_0^1 y\,dy\right] = \frac{\sigma}{\epsilon_0}\left[+\frac{1}{2}+\frac{1}{2}\right] = \frac{\sigma}{\epsilon_0}$

$m=2$ $\quad \frac{\sigma}{\epsilon_0}\left[-\int_{-1}^{0}\frac{3y^2-1}{2}dy + \int_0^1 \frac{3y^2-1}{2}dy\right] = \frac{\sigma}{\epsilon_0}\left[\left(\frac{-1}{2}+\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{2}\right)\right] = 0$

$m=3$ $\quad \frac{\sigma}{\epsilon_0}\left[-\int_{-1}^{0}\frac{5y^3-3y}{2}dy + \int_0^1 \frac{5y^3-3y}{2}dy\right] = \frac{\sigma}{\epsilon_0}\left[\left(\frac{5}{8}-\frac{3}{4}\right)+\left(\frac{5}{8}-\frac{3}{4}\right)\right] = \frac{-\sigma}{4\epsilon_0}$

$n=4$, hmm even m integrate to 0

$m=5$ $\quad \frac{\sigma}{\epsilon_0}\left[\int_0^{-1}\frac{63y^5-70y^3+15y}{8}dy + \int_0^1 \frac{63y^5-70y^3+15y}{8}dy\right]$

$= \frac{\sigma}{8\epsilon_0}$

$m=6$ $\quad = 0$, again, by symmetry

So, the solution becomes

$$B_1 = \frac{\sigma}{\epsilon_0}\left(\frac{R^3}{2}\right)$$

$$B_3 = -\frac{\sigma}{4\epsilon_0}\left(\frac{R^5}{2}\right)$$

$$B_5 = \frac{\sigma}{8\epsilon_0}\left(\frac{R^7}{2}\right)$$

$$B_7 = \cdots$$

only odd powers, but they don't exactly go away

plugged into

$$\Phi = \begin{cases} \displaystyle\sum_{\substack{l=1 \\ \text{odd}}}^{N} \frac{B_l}{r^{l+1}} P_l(\mu) & r > R \\ \displaystyle\sum_{\substack{l=1 \\ \text{odd}}}^{N} \frac{B_l}{R^{2l+1}} r^l P_l(\mu) & r < R \end{cases}$$

<u>note</u>: for $r > R$ series, begins w/ $l = 1$

$\Rightarrow$ no monopole term.