Prob 3.23

Solve Laplace's Eqn. by separation of Variables

$$\nabla^2 V = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right] V = 0$$

let $V = R(r)\Phi(\phi)$

$$\nabla^2 V = \left\{\frac{\Phi(\phi)}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} R(r) + \frac{R(r)}{r^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}\right] = 0$$

$$\Rightarrow \frac{1}{R(r)\, r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + \frac{1}{\Phi(\phi)\, r^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} = 0$$

multiply by r^2

$$\underbrace{\frac{r}{R(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)}_{+m^2} + \underbrace{\frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}}_{-m^2} = 0$$

(A) and $\frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} = -m^2 \Rightarrow \frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + m^2 \Phi(\phi) = 0$

$$\boxed{\Phi(\phi) = A\cos m\phi + B\sin m\phi}$$

(B) $\frac{r}{R(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) = +m^2$

let $R(r) = C r^{\alpha}$

$$\frac{r}{R(r)}\frac{\partial}{\partial r}\left[\alpha C r^{\alpha}\right] = m^2$$

$$C\alpha^2 r^{\alpha} = m^2 C r^{\alpha} \Rightarrow \alpha^2 = m^2 \rightarrow \alpha = \pm m$$

and $\boxed{R(r) = C r^{m} + D r^{-m}}$

(C) Consider case when $m=0$

(i) $\dfrac{\partial^2 \Phi(\phi)}{\partial \phi^2} = 0 \Rightarrow \boxed{\Phi(\phi) = a\phi + b}$

(ii) $\dfrac{r}{R(r)} \dfrac{\partial}{\partial r}\left(r \dfrac{\partial R(r)}{\partial r}\right) = 0 \Rightarrow r \dfrac{\partial R(r)}{\partial r} = c$

$$\boxed{R(r) = c \ln r + d}$$

(D) Total solution is

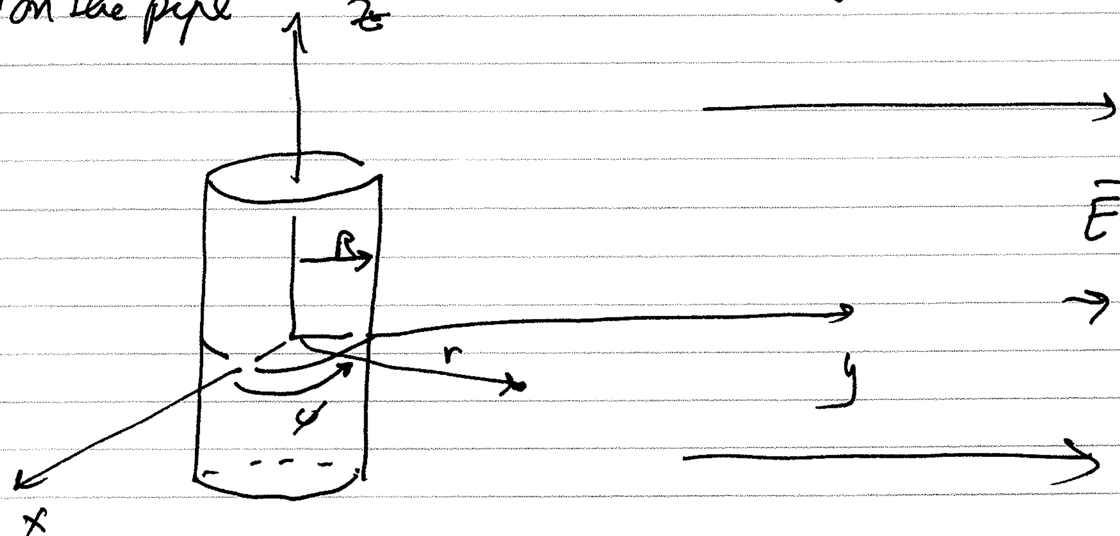
$$V(r,\phi) = \cancel{[c \ln r + d] + C r^{m} + D r^{-m}}$$
$$\boxed{(c \ln r + d)(a\phi + b) + (C r^{m} + D r^{-m})(A \cos m\phi + B \sin m\phi)}$$

(E) General Solution is then

$$V(r,\phi) = (a\phi + b)(c \ln r + d) + \sum_{m=1}^{\infty} \left(C_m r^{m} + \frac{D_m}{r^{m}}\right)\left(A_m \cos m\phi + B_m \sin m\phi\right)$$

Prob 3.24

Find the potential outside an infinitely long metal pipe of radius R placed at right ∡ to an otherwise uniform field E_0. Find σ on the pipe



$\vec{E} = E_0 \hat{y}$ at ∞

$\rightarrow V = -E_0 y$

$= -E_0 (r \sin\phi)$

a) Suppose $V(r=R,\phi) = V_0$. First let's consider the case where $V_0 = 0$

(i) let's require that $V(r=R,\phi) = 0$ by balancing $V(r,\phi)$

(ii) at $y \to \pm\infty$

$$V(r,\phi) = (Cr^m + Dr^{-m})(A\cancel{\cos m\phi} + B\sin m\phi) \quad \leftarrow \text{don't need log term}$$

and $m=1$

$$V(r,\phi) = \left(Cr + \frac{D}{r}\right)B\sin\phi \Rightarrow \boxed{C = -E_0}$$

(iii) at $r=R$, $V(r=R,\phi)=0$

$$\Rightarrow BCR + \frac{BD}{R} = 0 \Rightarrow C = -\frac{D}{R^2}$$

$$V(r,\phi) = D\left(-\frac{r}{R^2} + \frac{1}{r}\right)\sin\phi$$

$$= -E_0 R^2\left(-\frac{r}{R^2} + \frac{1}{r}\right)\sin\phi$$

$$\boxed{V(r,\phi) = E_0\left(r - \frac{R^2}{r}\right)\sin\phi}$$

(2)

b) Find $\sigma(\phi)$

$$E_r = -\frac{\partial}{\partial r} V(r,\phi)\Big|_{r=R} = -E_0\left(1+\frac{R^2}{r^2}\right)\sin\phi\Big|_{r=R}$$

$$\Rightarrow \sigma(\phi) = -\epsilon_0 E_0\left(1+\frac{R^2}{R^2}\right)\sin\phi$$

$$\boxed{\sigma(\phi) = -2\epsilon_0 E_0 \sin\phi}$$

c) Suppose cylinder is not grounded so that

$V(r=R,\phi) = V_0(\phi) \neq 0.$

We then need to add a sol^{n} where $V_{ins}(r=R,\phi) = V_0(\phi)$ and $V_{ins} \to 0$ as $\sqrt{x^2+y^2} \to \infty$

$\Rightarrow A_\ell^{ins} = 0$ for all ℓ

and

B_ℓ are defined by $V_0(\phi)$.

For example, if $V_0 =$ ~~constant~~ $V_0 \sin\phi$

$$\Rightarrow V_{ins}(r=R,\phi) = \cancel{\sum_{\ell=0}^{\infty} \frac{B_\ell}{R^\ell}}$$

$$= \sum_{m=0}^{\infty} \frac{B_m}{R^m} \sin m\phi$$

$\rightarrow B_1 = V_0 R$, $B_m = 0$ for $m \neq 1$

$$V_{ins}(r,\phi) = \frac{V_0 R}{r}\sin\phi \longrightarrow 0 \text{ as } r \to \infty$$

Prob 3.26

A sphere of radius R has

$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)\sin\theta$$

Find the approximate potential on the z-axis for $r >> R$

Sol^n

(1) Monopole: $Q = \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' \sin\theta' d\theta' \int_0^{R} k\frac{R}{r'^2}(R-2r')r'^2 dr'$

$$= 2\pi \times \pi/2 \times (kR^3 - kR^3)$$

$$= 0$$

(2) Dipole: $\vec{p} = \hat{z}\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' \sin\theta' d\theta' \cos\theta' \int \frac{kR}{r'^2}(R-2r')r'^2 dr \; r'\cos\theta'$

$$= \hat{z}(2\pi)\left(\frac{\sin^3\theta'}{3}\right)\Big|_0^{\pi}\left(k\frac{R^4}{2} - k\frac{2}{3}R^4\right)$$

$$= 0$$

(3) Quadrupole: $Q_{uad} = \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' \sin\theta' d\theta' P_2(\theta') \int \frac{kR}{r'^2}(R-2r') r'^2 r'^2 dr'$, with $P_2(\theta') = \frac{3}{2}\cos^2\theta' - \frac{1}{2}$

$$\int_0^{\pi} \sin\theta' \sin\theta' \left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\theta' \;{}^{*} = -\frac{\pi}{16}$$

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* note: $\int \cos^m x \sin^n x\, dx = -\frac{\sin^{n-1}x \cos^{m+1}x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x\, dx$

e

$$\text{Quad} = -2\pi \times \frac{\pi}{16} \times \left(k\frac{R^5}{3} - k\frac{R^5}{2} \right)$$

$$= + k \frac{R^5}{3\cancel{6}} \cancel{2}\pi \times \frac{\pi}{16}$$

$$= + k \frac{\pi^2 R^5}{48}$$

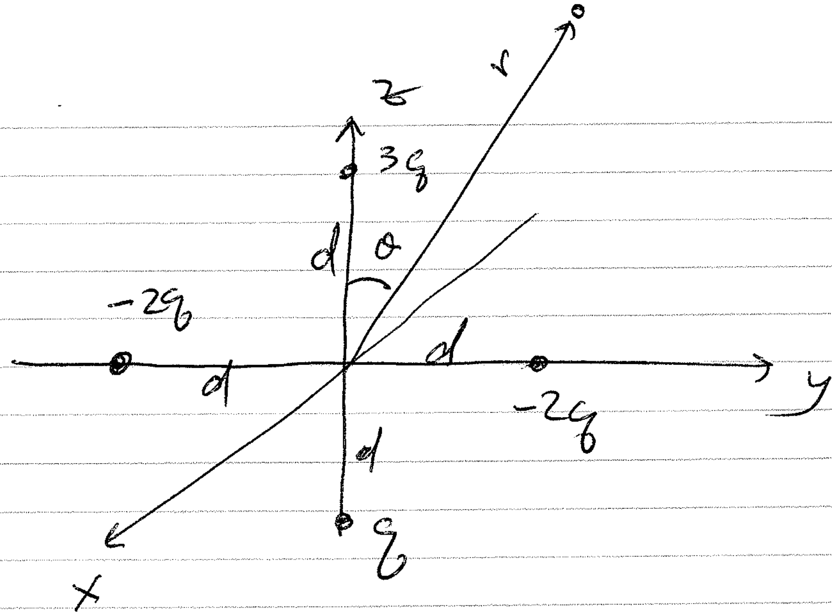
$$\Rightarrow V(r,\theta) = \frac{1}{4\pi\epsilon_0} \left(\frac{k\pi^2 R^5}{48} \right) \frac{P_2(\cos\theta)}{r^3}$$

Proh 3.28

Find the moments

① $Q = 0$

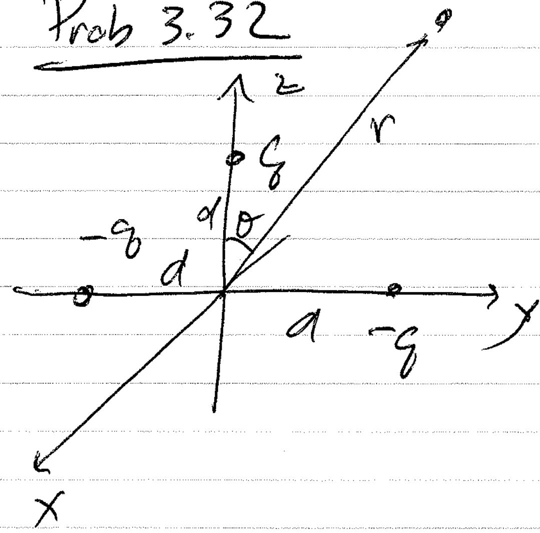
② $\vec{p} = \hat{z}\left[2qd\right]$



for $r \gg d$

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{2qd}{r^2}\cos\theta$$

Prob 3.32



Find $\vec{E}(r,\theta)$ for $r >> d$

① find moments, $Q \equiv$ total charge

$Q = -q$

② $\vec{p} = qd\hat{z} - qd\hat{y} - q(-d)\hat{y}$

$= qd\hat{z}$

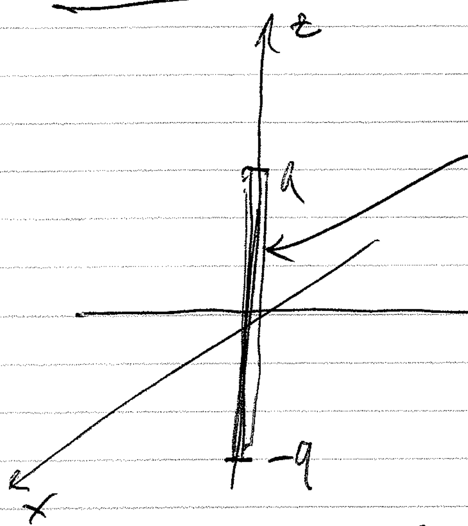
$$\Rightarrow V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{-q}{r}\right) + \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$$

$$\vec{E}(r,\theta) = -\vec{\nabla}V(r,\theta)$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} + \frac{1}{4\pi\epsilon_0}\frac{2qd\cos\theta}{r^3}\hat{r}$$

$$+ \frac{1}{4\pi\epsilon_0}\frac{qd\sin\theta}{r^3}\hat{\theta}$$

Problem 3.40



$$\lambda = \begin{cases} \lambda_0 \cos\left(\frac{\pi z}{2a}\right) & (a) \\ \lambda_0 \sin\left(\frac{\pi z}{2a}\right) & (b) \\ \lambda_0 \cos\left(\frac{\pi z}{a}\right) & (c) \end{cases}$$

Find the leading term in the multipole expansion for each λ

a) $$Q = \int_{-a}^{a} \lambda_0 \cos\left(\frac{\pi z}{2a}\right) dz = +\frac{2a}{\pi} \lambda_0 \sin\left(\frac{\pi z}{2a}\right)\Big|_{-a}^{a}$$

$$= \frac{2a\lambda_0}{\pi}\left[1-(-1)\right]$$

$$= \frac{4a\lambda_0}{\pi}$$

$$\Rightarrow V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{4a\lambda_0}{\pi r}\right)$$

b) $$Q = \int_{-a}^{a} \lambda_0 \sin\left(\frac{\pi z}{a}\right) dz = -\frac{a\lambda_0}{\pi} \cos\left(\frac{\pi z}{a}\right)\Big|_{-a}^{a}$$

$$= -\frac{a\lambda_0}{\pi}\left[-1-(-1)\right] = 0$$

$$\vec{p} = \hat{z}\int_{-a}^{a} \lambda_0 \sin\left(\frac{\pi z}{a}\right) z\, dz = \hat{z}\lambda_0\left(\frac{a}{\pi}\right)^2\left[\sin\frac{\pi z}{a} - \frac{\pi z}{a}\cos\frac{\pi z}{a}\right]_{-a}^{a}$$

$$= \hat{z}\lambda_0\left(\frac{a}{\pi}\right)^2\left[0-\left(\pi(-1)-(-\pi)(-1)\right)\right]$$

$$= \hat{z}\lambda_0\left(\frac{a}{\pi}\right)^2\left[2\pi\right]$$

②

$$\vec{p} = \hat{z}\,\lambda_0 \frac{2a^2}{\pi}$$

$$\Rightarrow V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{2\lambda_0 a^2}{\pi}\right)\frac{\cos\theta}{r}$$

c) $$Q = \int_{-a}^{a} \lambda_0 \cos\left(\frac{\pi z}{a}\right) dz$$

$$= \frac{a}{\pi}\lambda_0 \sin\left(\frac{\pi z}{a}\right)\Big|_{-a}^{a}$$

$$= 0$$

$$\vec{p} = \hat{z}\int \lambda_0 \cos\left(\frac{\pi z}{a}\right) z\,dz$$

$$= \hat{z}\,\lambda_0\left(\frac{a}{\pi}\right)^2\left[\cos\left(\frac{\pi z}{a}\right) + \left(\frac{\pi z}{a}\right)\sin\left(\frac{\pi z}{a}\right)\right]_{-a}^{a}$$

$$= \hat{z}\,\lambda_0\left(\frac{a}{\pi}\right)^2\left[-1-(-1) + \pi(0) - (-\pi)(0)\right]$$

$$= 0$$

$$\text{Quad} = Q_{zz} = \int_{-a}^{a} (3z^2 - z^2)\,\lambda_0 \cos\left(\frac{\pi z}{a}\right) dz$$

$$= \int_{-a}^{a} \lambda_0\, 2z^2 \cos\left(\frac{\pi z}{a}\right) dz$$

$$= 2\lambda_0\left(\frac{a}{\pi}\right)^3\left[2\,\frac{\pi z}{a}\cos\frac{\pi z}{a} + \left(\frac{\pi^2 z^2}{a^2} - 2\right)\sin\frac{\pi z}{a}\right]_{-a}^{a}$$

$$= 2\lambda_0\left(\frac{a}{\pi}\right)^3\left[2\pi(-1) - (-2\pi)(-1) + (\pi^2-2)0 - (\pi^2-2)0\right]$$

$$= -8\pi\lambda_0\left(\frac{a^3}{\pi^3}\right) = -8\lambda_0\frac{a^3}{\pi^2}$$

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(-\frac{8\lambda_0 a^3}{\pi^2}\right)\frac{P_2(\cos\theta)}{r^2}$$