Test 2

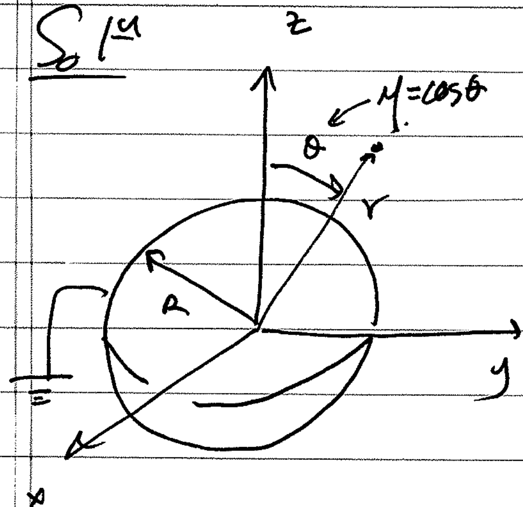
Problem 1

A grounded conducting sphere of radius R is placed into an otherwise uniform electric field.

a) Find V everywhere

b) Find σ on the conductor

Soln



Boundary Conditions

a) as $r \to \infty$, $\vec{E} = E_0 \hat{z}$

$$\Rightarrow V = -E_0 z + V_\infty$$

$$= -E_0 r\mu + V_\infty$$

b) $V = 0$ at $r = R$

the general solution is

$$V(r,\mu) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\mu)$$

(A) Apply BC at $r = \infty \Rightarrow A_1 = -E_0$; $A_\ell = 0$, $\ell \neq 1$

(B) at $r = R$, $V(r=R,\mu) = 0 = \sum_{\ell=0}^{\infty} \left(\frac{B_\ell}{R^{\ell+1}} P_\ell(\mu) \right) - E_0 R\mu + V_\infty$

(i) $\ell = 0 \Rightarrow \frac{B_0}{R} + V_\infty = 0 \Rightarrow B_0 = -V_\infty R$

$\ell = 1 \Rightarrow \frac{B_1}{R^2} P_1 - E_0 R P_1 = 0 \Rightarrow B_1 = E_0 R^3$

$\ell = 2 \Rightarrow \frac{B_2}{R^3} P_2 = 0 \Rightarrow B_2 = 0$

and by analogy $B_\ell = 0$ if $\ell > 2$

$$\Rightarrow \boxed{V(r,\mu) = \left(1-\frac{R}{r}\right)V_\infty + \left(-E_0 r + \frac{E_0 R^3}{r^2}\right)P_1(\mu)}$$

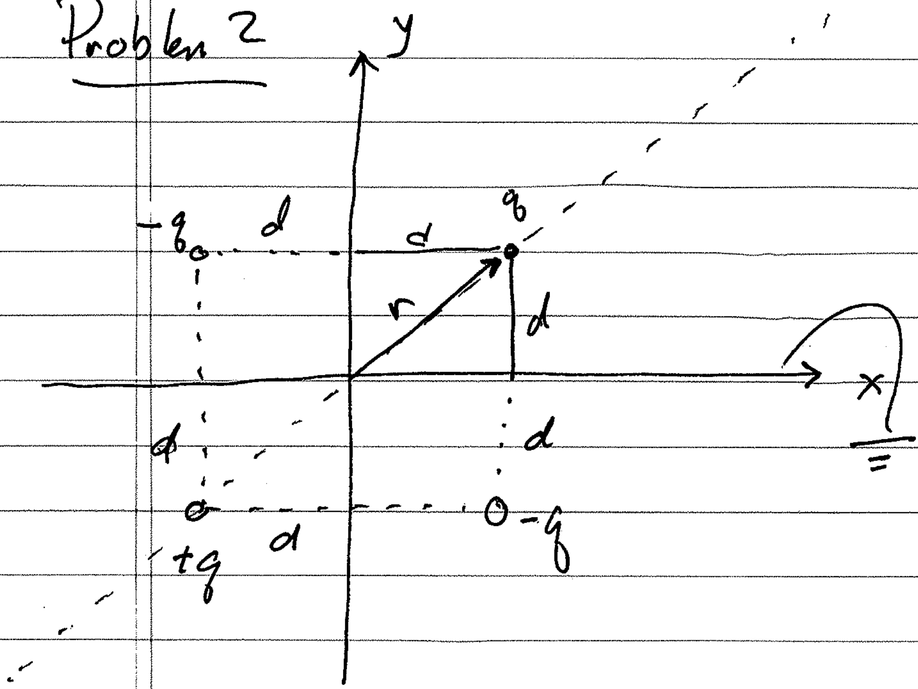
Q: what is V_∞?

(c) Find σ, $E_r = \sigma/\epsilon_0$ at $r=R$

$$\frac{\sigma}{\epsilon_0} = -\left[+\frac{R}{R^2}V_\infty - E_0\mu - \frac{2E_0R^3}{R^3}\mu\right]$$

$$\boxed{\sigma = 3\epsilon_0 E_0 \mu - \epsilon_0 \frac{V_\infty}{R}}$$

Problem 2



a) need 3 image charges as shown to the left.

$$\vec{F}_q = \hat{x}\left[\frac{-q^2}{4\pi\epsilon_0}\frac{1}{(2d)^2}\right] + \hat{y}\left[\frac{-q^2}{4\pi\epsilon_0}\frac{1}{(2d)^2}\right] + \frac{q^2}{4\pi\epsilon_0}\frac{1}{(8d^2)}\left(\hat{x}\cos\frac{\pi}{4} + \hat{y}\sin\frac{\pi}{4}\right)$$

b) Find work needed to bring in q from ∞

$$\vec{F}_q = \hat{x}\left(\frac{1}{4d^2} - \frac{1}{8d^2}\cos\frac{\pi}{4}\right)\left(\frac{-q^2}{4\pi\epsilon_0}\right) + \hat{y}\left(\frac{1}{4d^2} - \frac{1}{8d^2}\sin\frac{\pi}{4}\right)\left(-\frac{q^2}{4\pi\epsilon_0}\right)$$

Bring in the charge along the path which starts at

$(x,y) = (\infty,\infty)$

and ends at

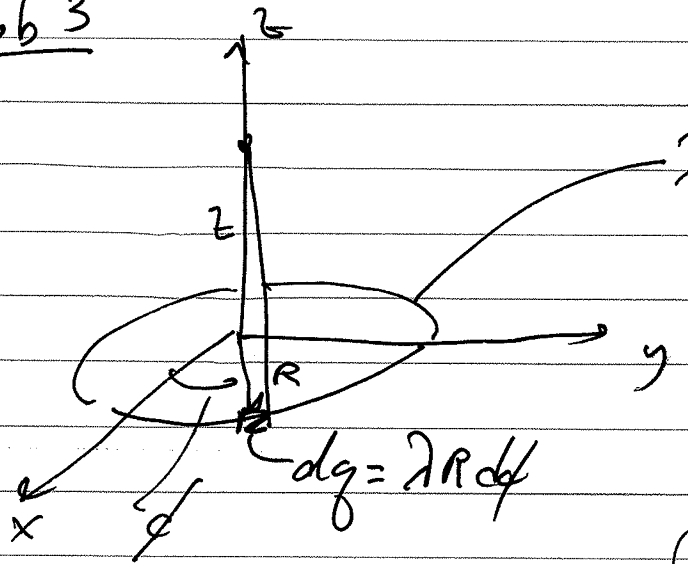
$(x,y) = (d,d)$

along the dotted line.

$$\vec{F}_q = \hat{r}\left[\frac{q^2}{4\pi\epsilon_0}\frac{1}{(2r)^2} + 2\frac{-q^2}{4\pi\epsilon_0}\frac{1}{(\sqrt{2}r)^2}\frac{\sqrt{2}}{2}\right]$$

$$W = -\frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{4}\int_\infty^d \frac{dr}{r^2} - \frac{\sqrt{2}}{2}\int_\infty^d \frac{dr}{r^2}\right) = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{4d} - \frac{\sqrt{2}}{2d}\right)$$

Prob 3



a) Find V on axis

$$V = \int \frac{1}{4\pi\epsilon_0} \frac{\lambda R\, d\phi}{\sqrt{R^2+z^2}}$$

$$\boxed{V = \frac{1}{2\epsilon_0} \frac{\lambda R}{\sqrt{R^2+z^2}}}$$

b) find $\vec{E}$ in plane of loop ($z=0$) for $r<R$. Use

$\cancel{\vec{E} \approx -\vec{\nabla}V =}$ $V(r,\mu) = \sum_{l=0} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\mu)$ (i)

to find $V(r,\mu)$ everywhere to get $\vec{E}$.

c) Expand $V(z)$ on axis

$$\Rightarrow V(z) = \frac{\lambda R}{2\epsilon_0} \frac{1}{R} \left(1 - \frac{1}{2}\left(\frac{z}{R}\right)^2 - \frac{1}{2}\left(-\frac{3}{2}\right)\frac{1}{2!}\left(\frac{z}{R}\right)^4 - \frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{3!}\left(\frac{z}{R}\right)^6 + \cdots\right.$$

$$\approx \frac{\lambda}{2\epsilon_0}\left(1 - \frac{1}{2}\left(\frac{z}{R}\right)^2 + \frac{3}{8}\left(\frac{z}{R}\right)^4 - \frac{5}{16}\left(\frac{z}{R}\right)^6\right) \quad \text{(ii)}$$

Compare (i) & (ii) $\Rightarrow$ $A_0 = \frac{\lambda}{2\epsilon_0}$, $A_1 = 0$, $A_2 = -\frac{\lambda}{4\epsilon_0 R^2}$, $A_3 = 0$,

$$A_4 = +\frac{3\lambda}{16\epsilon_0 R^4}$$

$$\underset{r<R}{V(r,\mu)} \approx \frac{\lambda}{2\epsilon_0} - \frac{\lambda}{4\epsilon_0 R^2}\left(r^2 P_2(\mu)\right) + \frac{3\lambda}{16\epsilon_0 R^4}\left(r^4 P_4(\mu)\right)$$

d) $\underline{\text{find } \vec{E} = -\vec{\nabla} V \text{ for } \mu = 0}$

$$\vec{E} = \frac{\lambda}{4\epsilon_0 R^2}\left(2r P_2(\mu)\hat{r} - \frac{r^2}{r}\frac{\partial}{\partial\theta}\left[\frac{3}{2}\cos^2\theta - \frac{1}{2}\right]\hat{\theta}\right)$$

0 when $\theta = \frac{\pi}{2}$

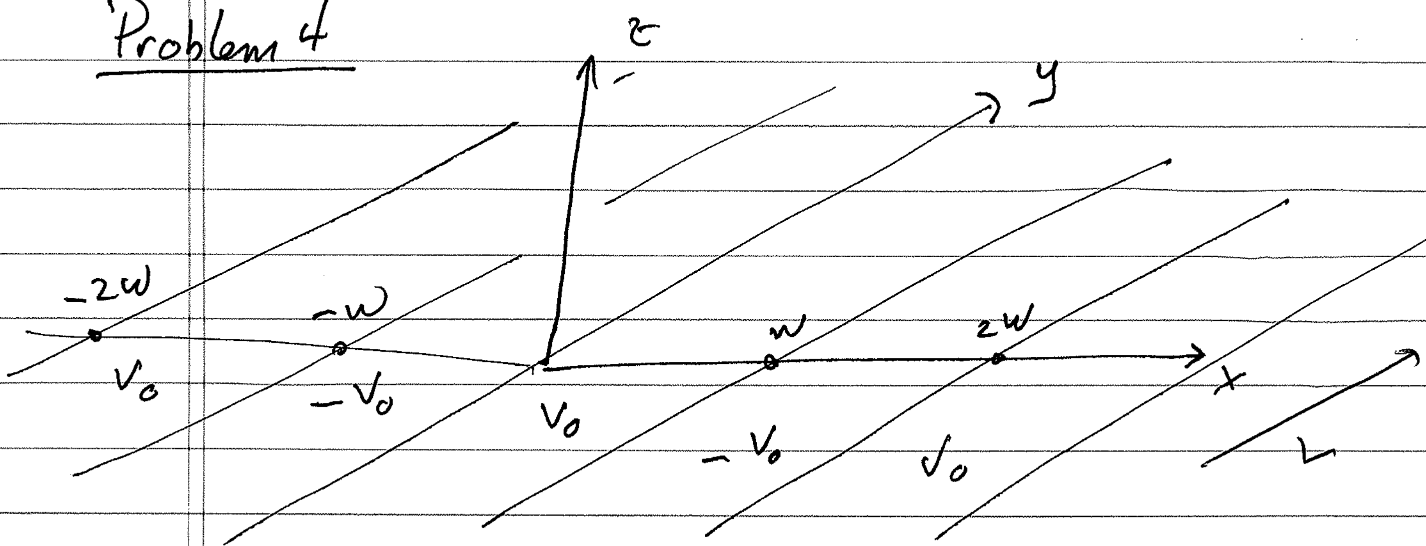
$$+ \frac{3\lambda}{16\epsilon_0 R^4}\left(4r^3 P_4(\mu)\hat{r} - \frac{r^4}{r}\frac{\partial}{\partial\theta}\left[\frac{35\cos^4\theta - 30\cos^2\theta + 3}{8}\right]\hat{\theta}\right)$$

0 when $\theta = \frac{\pi}{2}$

$$\boxed{\vec{E} = \frac{\lambda}{2\epsilon_0 R^2} r P_2(\mu)\hat{r} + \frac{3\lambda}{4\epsilon_0 R^4} r^3 P_4(\mu)\hat{r}}$$

$P_2(\mu = 0) = -\frac{1}{2}$ $\qquad$ $P_4(\mu = 0) = +\frac{3}{8}$

Problem 4



(i) for $z=0$, $V = \left\{ \begin{matrix} V_0 \\ -V_0 \end{matrix} \right\}$ in steps of width w as $x \to \pm\infty$

(ii) for $z \to \pm\infty$, $V \to 0$

(iii) at $\pm wn = x$, $V \to 0$, by symmetry

"Base" solution

$$V(x,y,z) = (A\cos\alpha x + B\sin\alpha x)(C\cos\beta y + D\sin\beta y) \times (Ee^{\gamma_{\alpha\beta}z} + Fe^{-\gamma_{\alpha\beta}z})$$

(i) as $z \to \infty \Rightarrow V \to 0 \Rightarrow E = 0$

(ii) Because the "slats" extend to $\pm y$ $(L=\infty)$
$\Rightarrow$ must be independent of y
$\Rightarrow \beta = 0$

okay, so we have that

$$V(x,z) = (A\cos\alpha x + B\sin\alpha x)\,C\cancel{F}e^{-\gamma_\alpha z}$$

(iii) let's match the periodic boundary condition. Because

$$V = 0 \text{ at } x=0 \text{ \& } x = 2W$$

consider a "slot."

(iv) $V=0$ at $x=0 \Longrightarrow A=0$

$V=0$ at $x=2W \Longrightarrow \alpha = \frac{m\pi}{2W}$, $m=\cancel{0},1,2,3,\ldots$

and we have

$$V(x,z) = \underbrace{B_m C \cancel{F}}_{C_m} \sin\left(\frac{m\pi}{2W}x\right)e^{-\frac{m\pi}{2W}z}$$

(v) Match BC for $z=0$ using Fourier series

$$V(x,z=0) = \sum_{m=1}^{\infty} C_m \sin\left(\frac{m\pi}{2W}x\right)$$

multiply through by $\sin\left(\frac{k\pi}{2W}x\right)dx$ and integrate

(vi) $$\int_0^W V_0 \sin\left(\frac{k\pi}{2W}x\right)dx - \int_W^{2W} V_0 \sin\left(\frac{k\pi}{2W}x\right)dx$$

$$= \sum_{m=1}^{\infty} C_m \int_0^{2W} \sin\left(\frac{m\pi}{2W}x\right)\sin\left(\frac{k\pi}{2W}x\right)dx$$

$$= \sum_{m=1}^{\infty} W C_m \delta_{km}$$

Set $k = m$

$$\frac{2wV_0}{m\pi}\left[-\cos\left(\frac{m\pi}{2w}x\right)\Big|_0^{w} + \cos\left(\frac{m\pi}{2w}x\right)\Big|_w^{2w}\right] = w\,C_m$$

$$\Rightarrow C_m = \frac{2V_0}{m\pi}\left[-\cos\left(\frac{m\pi}{2}\right) + 1 + \cos(m\pi) - \cos\left(\frac{m\pi}{2}\right)\right]$$

$$= \frac{2V_0}{m\pi}(4);\ \text{if } m = 2, 6, 10, \ldots$$

$$\Rightarrow V(x,z) = \frac{8V_0}{\pi}\sum_{m=2,6,10,\ldots}\frac{1}{m}\sin\left(\frac{m\pi}{2w}x\right)e^{-\left(\frac{m\pi}{2w}z\right)}$$

and then solution repeats in each pair of slots