HW #3

5.31

a) $\vec{\nabla} \cdot \vec{F} = 0$ & $\vec{F} = \vec{\nabla} \times \vec{A}$

(i)
$$F_x = \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y \quad ①$$
$$F_y = \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z \quad ②$$
$$F_z = \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \quad ③$$

① $\frac{\partial}{\partial x} F_x = \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z}$

② $\frac{\partial}{\partial y} F_y = \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x}$

③ $\frac{\partial}{\partial z} F_z = \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$

add ① + ② + ③ $\Rightarrow$ $\underbrace{\frac{\partial}{\partial x} F_x + \frac{\partial F_y}{\partial x} + \frac{\partial}{\partial z} F_z}_{\vec{\nabla} \cdot \vec{F} = 0} = 0$

b)

Find $\vec{A}$ in terms of integrals of $\vec{F}$

(i) Let $A_x = 0$ for a solution

② $\Rightarrow$ $F_y = -\frac{\partial}{\partial x} A_z \Rightarrow A_z = -\int_0^x F_y(x', y, z)\, dx' + G(y,z)$

③ $\Rightarrow$ $F_z = \frac{\partial}{\partial x} A_y \Rightarrow A_y = \int_0^x F_z(x', y, z)\, dx' + H(y,z)$

Now for ①

$$F_x = \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y$$
$$= -\int_0^x \frac{\partial F_y}{\partial y} dx' + \frac{\partial G}{\partial y} - \int_0^x \frac{\partial F_z}{\partial z} dx' - \frac{\partial H}{\partial z}$$

note: $\vec{\nabla}\cdot\vec{F} = 0$

$$\Rightarrow \underbrace{\int_0^x dx'\left[\frac{\partial F_x}{\partial x'}\right]} + \frac{\partial G}{\partial y} - \frac{\partial H}{\partial z} = F_x(x,y,z)$$

$$F_x(x,y,z) - F_x(0,y,z) + \frac{\partial G}{\partial y} - \frac{\partial H}{\partial z} = F(x,y,z)$$

$$\text{and} \quad -F_x(0,y,z) = \frac{\partial H}{\partial z} - \frac{\partial G}{\partial y}$$

set our BCs by taking $H = 0$

$$\Rightarrow +\frac{\partial G}{\partial y} = F_x(0,y,z)$$

$$\Rightarrow +G = \int_0^y F_x(0,y',z)\,dy'$$

$$G = +F_x(0,y,z)\,F_x(0,0,z)$$

$$\Rightarrow \boxed{\begin{aligned} &A_x = 0,\ A_y = \int_0^x F_z(x',y,z)\,dx', \\ &A_z = -\int_0^x F_y(x',y,z)\,dx' + F_x(0,y,z) \\ &\qquad F_x(0,0,z) \end{aligned}}$$

b) find $\vec{\nabla} \times \vec{A}$

$$= \left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y , \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z , \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \right)$$

$$= \left(-\int_0^x \frac{\partial F_y}{\partial y} dx' + F_x(0,y,z) - \int_0^x \frac{\partial F_z}{\partial z} dx' , \right.$$

$$\left. 0 - \left[-F_y(x,y,z) \right] , F_z(x,y,z) \right)$$

use $\vec{\nabla} \cdot \vec{F} = 0$ in $\hat{x}$-term

$$= \left(+\int_0^x \frac{\partial F_x}{\partial x'} dx' + F_x(0,y,z) , F_y(x,y,z) , F_z(x,y,z) \right)$$

$$= \left(F_x(x,y,z) - F_x(0,y,z) + F_x(0,y,z) , F_y(x,y,z) , F_z(x,y,z) \right)$$

$= \vec{F}$ as required

c) find $\vec{\nabla} \cdot \vec{A}$

$$\frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z = 0 + \int_0^x \frac{\partial F_z}{\partial y} dx' + \int_0^x \frac{\partial F_y}{\partial z} dx'$$

$$- \int_0^y \frac{\partial F_x(0,y',z)}{\partial z} dy'$$

$\neq 0$

d) $\vec{F} = (y, z, x)$

$$\Rightarrow A_x = 0 , A_y = \int_0^x x' dx' = x^2/2 , A_z = -xz + y^2/2$$

(i) $\vec{\nabla} \times \vec{A} = (+y - 0, 0 + z, x - 0)$

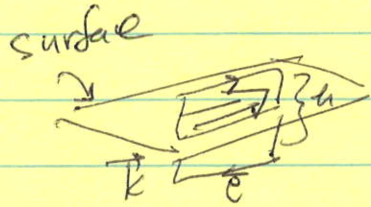
(ii) $\vec{\nabla} \cdot \vec{A} = 0 + 0 + (-x) \neq 0$

5.33

Prove $\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 K$

where $\hat{n}$ is a unit vector $\perp$ to the surface

(i) $\vec{A}_{above} = \vec{A}_{below}$ for gauge $\vec{\nabla}\cdot\vec{A} = 0$ and $\vec{\nabla}\times\vec{A} = \vec{B}$ has form $\oint \vec{A}\cdot d\vec{l} = \int \vec{B}\cdot d\vec{S}$

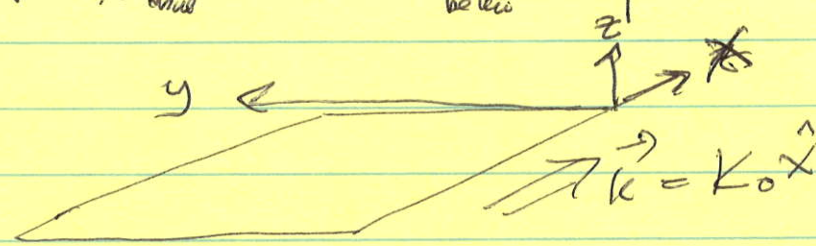


as $h \to 0$ $\int \vec{B}\cdot d\vec{S} \to 0$

(ii) We have

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K}\times\hat{n})$$

ⓐ $\vec{\nabla}\times\vec{A}_{above} - \vec{\nabla}\times\vec{A}_{below} = \mu \vec{K}\times\hat{n}$



ⓐ $\Rightarrow \Big(\frac{\partial}{\partial y}A_{z,above} - \frac{\partial}{\partial z}A_{y,above}, \frac{\partial}{\partial z}A_{x,above} - \frac{\partial}{\partial x}A_{z,above},$

$\frac{\partial}{\partial x}A_{y,above} - \frac{\partial}{\partial y}A_{x,above}\Big)$

$-\Big(\frac{\partial}{\partial y}A_{z,below} - \frac{\partial}{\partial z}A_{y,below}, \frac{\partial}{\partial z}A_{x,below} - \frac{\partial}{\partial x}A_{z,below}, \frac{\partial}{\partial x}A_{y,below} - \frac{\partial}{\partial y}A_{x,below}\Big)$

$= \mu_0 \vec{K}\times\hat{n}$

$\underline{B/C \quad \vec{A}_{above} = \vec{A}_{below}}$

$$\left(\underbrace{-\frac{\partial}{\partial z} A_{y,above} + \frac{\partial}{\partial z} A_{y,below}}_{-\mu_0 K}, \underbrace{\frac{\partial}{\partial z} A_{x,above} - \frac{\partial}{\partial z} A_{x,below}}_{0}, 0 \right) = \mu_0 \vec{K} \times \hat{n}$$

$\vec{K} = K_0 \hat{x}$

$$\boxed{\begin{aligned} &a) \quad \frac{\partial}{\partial z} A_{y,below} - \frac{\partial}{\partial z} A_{y,above} = -\mu_0 K \\ &b) \quad \frac{\partial}{\partial z} A_{x,below} - \frac{\partial}{\partial z} A_{x,above} = 0 \end{aligned}}$$

5.34

Show that the magnetic field can be written in coordinate free form

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$$

@ $\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\rightarrow \vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left[\frac{(\vec{m}\times\vec{r})}{r^3}\right]$$

$$= \frac{\mu_0}{4\pi}\left[\frac{1}{r^3}\vec{\nabla}\times(\vec{m}\times\vec{r}) - (\vec{m}\times\vec{r})\times\vec{\nabla}\frac{1}{r^3}\right]$$

note: $\vec{\nabla}\frac{1}{r^3} = \frac{1}{r^5}\left(-\frac{3}{2}2x, -\frac{3}{2}2y, -\frac{3}{2}2z\right) = -\frac{3\vec{r}}{r^5}$

$$= \frac{\mu_0}{4\pi}\left[\frac{1}{r^3}\vec{\nabla}\times(\vec{m}\times\vec{r}) - \frac{3\vec{r}}{r^5}\times(\vec{m}\times\vec{r})\right]$$

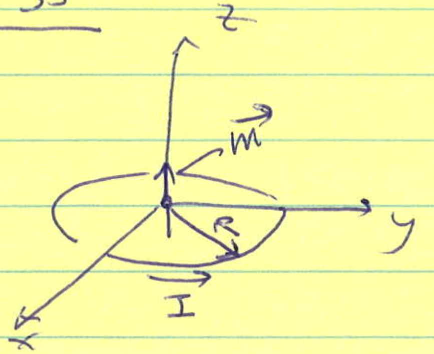
$$= \frac{\mu_0}{4\pi r^3}\left[\cancelto{0}{(\vec{r}\cdot\vec{\nabla})\vec{m}} - (\vec{m}\cdot\vec{\nabla})\vec{r} + \vec{m}(\vec{\nabla}\cdot\vec{r}) - \cancelto{0}{\vec{r}(\vec{\nabla}\cdot\vec{m})} - \frac{3\vec{r}}{r^2}\times(\vec{m}\times\vec{r})\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left[-\vec{m} + 3\vec{m} - \frac{3\vec{r}}{r^2}\times(\vec{m}\times\vec{r})\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left[2\vec{m} - \frac{3}{r^2}\left(\vec{m}(\vec{r}\cdot\vec{r}) - \vec{r}(\vec{m}\cdot\vec{r})\right)\right]$$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi r^3}\left[\frac{3\vec{r}(\vec{m}\cdot\vec{r})}{r^2} - \vec{m}\right]}$$

5.35



a) $$\boxed{\vec{m} = I\int d\vec{S} = I\pi R^2\hat{z}}$$

b) $$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}}$$

find $\vec{B} = \vec{\nabla}\times\vec{A}$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{\partial}{\partial r}rA_\phi\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial}{\partial\theta}A_r\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\mu_0 m}{4\pi r^2}2\sin\theta\cos\theta\right]\hat{r} + \frac{1}{r}\left[-\frac{\mu_0 m\sin\theta}{4\pi}\left(-\frac{1}{r^2}\right)\right]\hat{\theta}$$

$$\boxed{\vec{B} = \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}$$

c) on z axis, $\theta = 0 \rightarrow \sin\theta = 0$

$$\Rightarrow \vec{B} = \frac{\mu_0 m}{4\pi|z|^3}(2\hat{z})$$

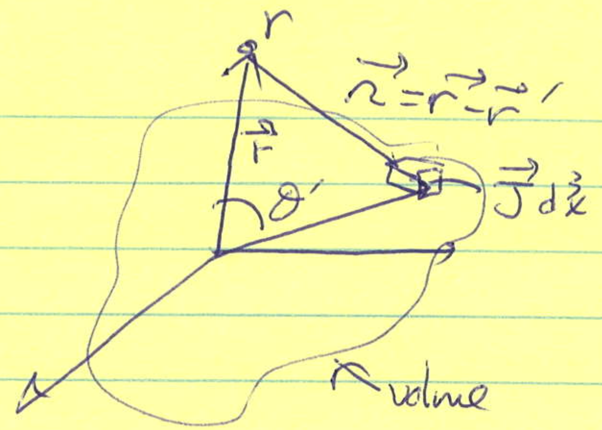
consistent to w/in $\mathcal{O}\left(\frac{z^2}{R^2}\right)$

Exact Solution:

$$\vec{B} = \frac{\mu_0 I\pi R^2}{2\pi(R^2+z^2)^{3/2}}\hat{z} \approx \frac{\mu_0 \overbrace{I\pi R^2}^{m}}{2\pi|z^3|}\left(1 - \frac{3}{2}\frac{z^2}{R^2}\right)\hat{z}$$

5.38

a) $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\,d^3x'}{\mathscr{r}}$



$$= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\,d^3x'}{\sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'}} \quad , |\vec{r}| > |\vec{r}\,'|$$

$$= \frac{\mu_0}{4\pi r}\int \frac{\vec{J}(\vec{r}')\,d^3x'}{\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{\vec{r}\cdot\vec{r}\,'}{r^2}}} \qquad \frac{\vec{r}\cdot\vec{r}\,'}{r^2} = \frac{r'}{r}\cos\theta'$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi r}\sum_{l=0}^{\infty}\int \vec{J}(\vec{r}')\,d^3x' \left(\frac{r'}{r}\right)^l P_l(\cos\theta')}$$

b) $l = 0$, monopole moment

$$\vec{A}_0 = \frac{\mu_0}{4\pi r}\int \vec{J}(\vec{r}')\,d^3x' = 0, \text{ current is enclosed in the volume}$$

c) find $\vec{m}$ by appropriate modification of

$$\vec{m} = \frac{1}{2} I \oint (\vec{r}\,' \times d\vec{l}) = \frac{1}{2}\oint (\vec{r}\,' \times I\,d\vec{l})$$

$$\Longrightarrow \boxed{\vec{m} = \frac{1}{2}\int \vec{r}\,' \times \vec{J}(\vec{r}')\,d^3x'}$$

(in class, I actually grunged this out to show how this comes about in a more rigorous manner)