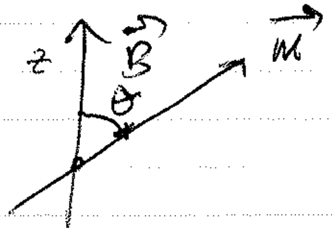
HW #5

6.21

a)



$$dW = -(\vec{m}\times\vec{B})\cdot(-d\theta\,\hat{z}) = mB\sin\theta\,d\theta$$

$$W = -mB\cos\theta$$

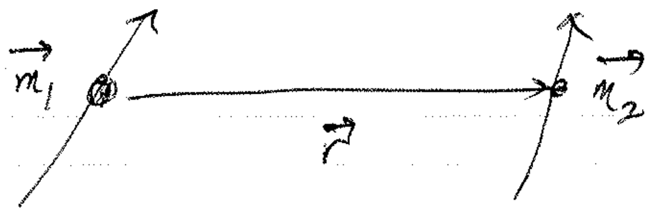
$$\rightarrow \boxed{U_B = -\vec{m}\cdot\vec{B}}$$

b) for two dipoles separated by $\vec{r}$ show that

$$U_B = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left(\vec{m}_1\cdot\vec{m}_2 - 3(\vec{m}_1\cdot\hat{r})(\vec{m}_2\cdot\hat{r})\right)$$

(Eq 5.89) $\hat{B}_{dip} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$

as shown in earlier HW assignment. so let

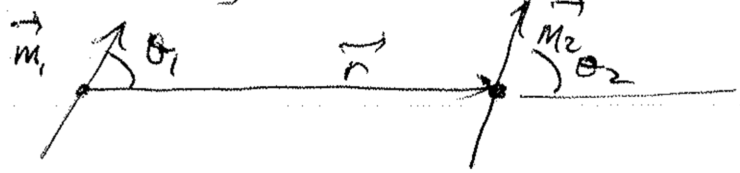


has $U_B = -\frac{\mu_0}{4\pi}\frac{1}{r^3}\vec{m}_2\cdot\left[3(\vec{m}_1\cdot\hat{r})\hat{r} - \vec{m}_1\right]$

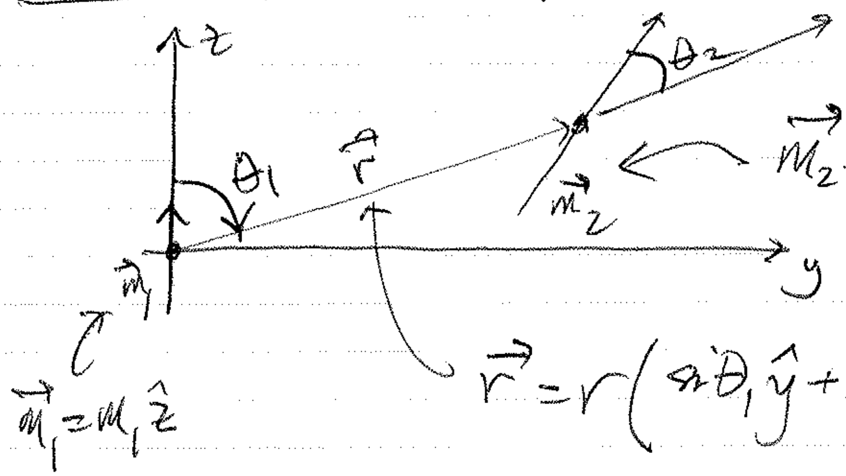
$$= -\frac{\mu_0}{4\pi r^3}\left[3(\vec{m}_1\cdot\hat{r})(\vec{m}_2\cdot\hat{r}) - \vec{m}_1\cdot\vec{m}_2\right]$$

$$\boxed{U_B = \frac{\mu_0}{4\pi r^3}\left[\vec{m}_1\cdot\vec{m}_2 - 3(\vec{m}_1\cdot\hat{r})(\vec{m}_2\cdot\hat{r})\right]}$$

c) Express your answer in part b using ∠s θ_1 & θ_2.



align $\vec{m}_1$ w/ z-axis & put $\vec{m}_1$ & $\vec{m}_2$ in y-z plane



$$\vec{m}_2 = m_2 \sin(\theta_1-\theta_2)\hat{y} + m_2\cos(\theta_1-\theta_2)\hat{z}$$

$$\vec{r} = r\left(\sin\theta_1\hat{y} + \cos\theta_1\hat{z}\right)$$

$$\Rightarrow U_B = \frac{\mu_0 m_1 m_2}{4\pi r^3}\left[\cos(\theta_1-\theta_2) - 3\cos\theta_1\left(\sin\theta_1\sin(\theta_1-\theta_2) + \cos\theta_1\cos(\theta_1-\theta_2)\right)\right]$$

$$= \frac{\mu_0 m_1 m_2}{4\pi r^3}\Big[\cos(\theta_1-\theta_2) - 3\cos\theta_1\Big\{\sin\theta_1\left(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1\right)$$
$$+ \cos\theta_1\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\Big\}\Big]$$

$$= \frac{\mu_0 m_1 m_2}{4\pi r^3}\left[\cos(\theta_1-\theta_2) - 3\cos\theta_1\left\{\cos\theta_2\right\}\right]$$

$$= \frac{\mu_0 m_1 m_2}{4\pi r^3}\left[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 - 3\cos\theta_1\cos\theta_2\right]$$

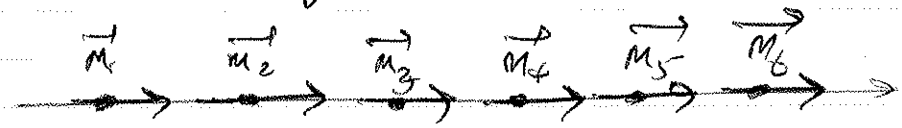
$$= \frac{\mu_0 m_1 m_2}{4\pi r^3}\left[\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2\right]$$

$$\Rightarrow \text{if } \theta_1 = \frac{\pi}{3},\ \theta_2 = \frac{3\pi}{2} \rightarrow U_B = -\frac{\mu_0 m_1 m_2}{4\pi r^3}$$
$(\Rightarrow \vec{N} = 0)$

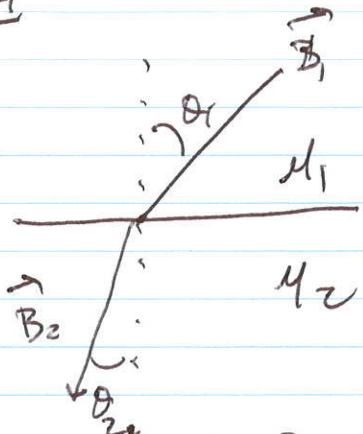
$$\theta_1 = 0,\ \theta_2 = 0 \rightarrow U_B = -\frac{\mu_0 m_1 m_2}{2\pi r^3}$$
$(\Rightarrow \vec{N} = 0)$

⇒ latter gives minimum energy state and preferred alignment.

d) would all align in the same direction.



6.27



show that $\dfrac{\tan\theta_2}{\tan\theta_1} = \dfrac{\mu_2}{\mu_1}$

no free surface curr

a) $\vec{\nabla}\cdot\vec{B} = 0 \Longrightarrow \Delta B_n = 0 \Longrightarrow \boxed{B_1\cos\theta_1 = B_2\cos\theta_2}$

b) $\vec{\nabla}\times\vec{H} = \vec{J}_f = 0 \Longrightarrow \Delta H_{\parallel} = 0 \Rightarrow H_{T,1} = H_{T,2}$

tangential field

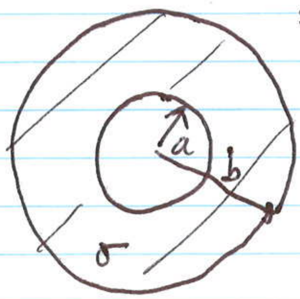
$\rightarrow \dfrac{1}{\mu_1} B_{T,1} = \dfrac{1}{\mu_2} B_{T,2}$

$\boxed{\mu_2 B_1 \sin\theta_1 = \mu_1 B_2 \sin\theta_2}$

$\dfrac{B_1}{B_2} = \dfrac{\cos\theta_2}{\cos\theta_1} = \dfrac{\mu_1 \sin\theta_2}{\mu_2 \sin\theta_1} \Longrightarrow \boxed{\dfrac{\tan\theta_2}{\tan\theta_1} = \dfrac{\mu_2}{\mu_1}}$

7.1

(a)



i) V is potential difference. Find the current I

ii) in steady state, $\vec{\nabla}\cdot\vec{J}=0$

$\rightarrow \vec{\nabla}\cdot\sigma\vec{E}=0 \rightarrow \vec{\nabla}\cdot\vec{E}=0$

and in steady state, $\vec{\nabla}\times\vec{E}=0$

$\rightarrow \vec{E}=-\vec{\nabla}V$

$$\Rightarrow \nabla^2 V=0 \rightarrow V=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

axisymmetric $\Rightarrow l=0$

and so $\boxed{V(r)=A_0+\frac{B_0}{r}}$

(iii) BCs, at $r=a$, $V=V \rightarrow V=A_0+\frac{B_0}{a}$

at $r=b$, $V=0 \rightarrow 0=A_0+\frac{B_0}{b} \rightarrow A_0=-\frac{B_0}{b}$

so $V=-\frac{B_0}{b}+\frac{B_0}{a}=B_0\left(\frac{1}{a}-\frac{1}{b}\right) \rightarrow B_0=\frac{V}{\frac{1}{a}-\frac{1}{b}}$

$A_0=-\frac{V}{\frac{b}{a}-1}$

$$\Rightarrow V(r)=-\frac{V}{\frac{b}{a}-1}+\frac{V}{\frac{1}{a}-\frac{1}{b}}\frac{1}{r}$$

and $\boxed{\vec{E}=-\vec{\nabla}V=+\frac{Vab}{b-a}\frac{1}{r^2}\hat{r}}$

$$\Rightarrow \vec{J}=\sigma V\frac{ab}{b-a}\frac{\hat{r}}{r^2}$$

and $\boxed{I=\sigma V\frac{ab}{b-a}4\pi}$

(b) $V = IR \Rightarrow \boxed{R = \frac{V}{I} = \left(\frac{b-a}{4\pi ab}\right)\frac{1}{\sigma}}$

(c) if $b \gg a \Rightarrow V(r) = V\frac{a}{r}$,

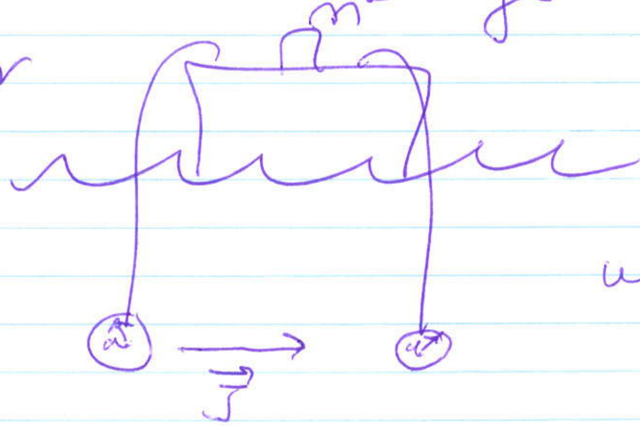
$$\vec{E}(r) = V\frac{a}{r^2}\hat{r},$$

$$I = \sigma V 4\pi a,$$

$$R = \frac{1}{4\pi a\sigma}$$

all independent of b. Most of the resistance comes where $\vec{J}$ is large near the smaller sphere.

For



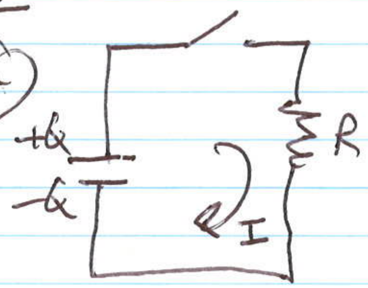
we have $R \simeq \frac{1}{4\pi a\sigma} + \frac{1}{4\pi a\sigma}$

$$= \frac{2}{4\pi a\sigma}$$

$$\boxed{R = \frac{1}{2\pi a\sigma}}$$

7.2

(a)



a capacitor has been charged to V_0 at $t=0$, it is connected to a resistor. find $Q(t)$.

(i) capacitor, $V = \dfrac{Q}{C} = IR = -R\dfrac{dQ}{dt}$

I comes from charge flowing off capacitor

$$\Rightarrow -\underbrace{\left[-R\frac{dQ}{dt}\right]}_{\text{Resistor}} - \underbrace{\frac{Q}{C}}_{\text{capacitor potential}} = 0$$

$$\boxed{\frac{dQ}{dt} + \frac{Q}{RC} = 0 \Rightarrow Q(t) = Q(0)e^{-\frac{t}{RC}}}$$

where $Q(0)$ is the charge on the capacitor at $t=0$

(b) $I = -\dfrac{dQ}{dt} = -\dfrac{Q(0)}{RC}e^{-\frac{t}{RC}}$

(i) the energy stored in a capacitor is $W = \frac{1}{2}CV^2 = \dfrac{Q^2}{2C}$

(ii) Integrate $P = IV = I^2R$

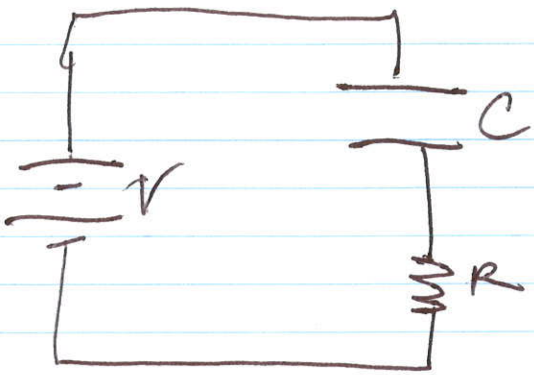
and $P = \left[-\dfrac{Q(0)}{RC}e^{-\frac{t}{RC}}\right]^2 R$

$$\int_0^\infty P\,dt = \frac{Q(0)^2}{(RC)^2}R\int_0^\infty e^{-\frac{2t}{RC}}dt$$

$$= \frac{Q(0)^2}{RC^2}\left(-\frac{RC}{2}\right)e^{-\frac{2t}{RC}}\Big|_0^\infty$$

$$\boxed{\int_0^\infty P\,dt = \frac{Q(0)^2}{2C} \quad \text{as asked}}$$

(c)



charge up the capacitor w/ the battery

we now have "battery" + "capacitor" + "resistor" = 0

$$\Rightarrow V = \frac{Q}{C} + IR = \frac{Q}{C} + \underbrace{R\frac{dQ}{dt}}_{Q\uparrow \text{ on capacitor}}$$

and $$\frac{dQ}{dt} + \frac{Q}{RC} = V/R$$

let: $$\psi = \frac{Q}{RC} - V/R$$

$$\Rightarrow \frac{d}{dt}\left[RC\left(\psi + \frac{V}{R}\right)\right] + \psi = 0$$

$$RC\frac{d\psi}{dt} + \psi = 0$$

$$\rightarrow \psi = \psi(0)\, e^{-\frac{t}{RC}}$$

and so,

$$\frac{Q}{RC} - \frac{V}{R} = \left(\frac{\cancel{Q(0)}^{\,0}}{RC} - \frac{V}{R}\right) e^{-\frac{t}{RC}}$$

$$Q = +\cancel{R}C\left[V - Ve^{-t/RC}\right]$$

$$\boxed{\Rightarrow Q(t) = CV\left(1-e^{-\frac{t}{RC}}\right) \Rightarrow I(t) = +\frac{CV}{RC}e^{-\frac{t}{RC}}}$$

(d) (i) $\int_0^\infty P_V\,dt = \int_0^\infty IV\,dt = +\int_0^\infty \frac{V^2}{R}e^{-\frac{t}{RC}}dt$

$$= -V^2Ce^{-t/RC}\Big|_0^\infty$$

$$= CV^2 \checkmark$$

(ii) $\int_0^\infty P_R\,dt = \int_0^\infty I^2R\,dt = \int_0^\infty \left(\frac{V}{R}\right)^2 e^{-\frac{2t}{RC}} R\,dt$

$$= \frac{V^2}{R^2}R\left(-\frac{RC}{2}\right)e^{-2t/RC}\Big|_0^\infty$$

$$= \frac{1}{2}CV^2 \checkmark$$

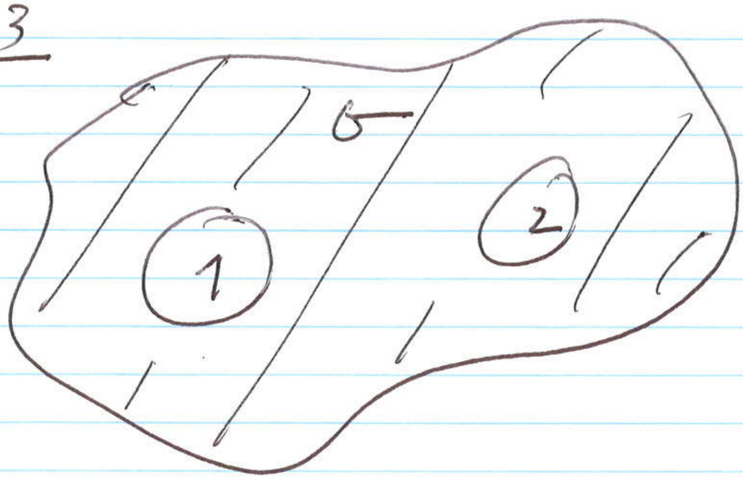
(iii) $Q(t) = CV\left(1-e^{-t/RC}\right)$ for capacitor

$\rightarrow$ as $t\rightarrow\infty$, $Q\rightarrow Q(\infty) = CV$

also, $W_C = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \checkmark$

(iv) $\boxed{\frac{W_C}{W_V} = \frac{1}{2}}$

7.3



show that $R = \frac{\epsilon_0}{\sigma C}$

a) on object 1, $Q_1 = \underbrace{\int \rho_1 \, d^3x = \oint \epsilon_0 \vec{E}_1 \cdot d\vec{S}}_{\text{Gauss's law}}$

$$I_1 = \oint \vec{J}_1 \cdot d\vec{S} = \oint \sigma \vec{E}_1 \cdot d\vec{S}$$

equate $\oint \vec{E}_1 \cdot d\vec{S}$ terms $\Longrightarrow \boxed{\frac{Q_1}{\epsilon_0} = \frac{I_1}{\sigma}}$

now, $\frac{Q_1}{\epsilon_0} = \frac{CV_1}{\epsilon_0} = \frac{C I_1 R}{\epsilon_0}$ which is equal to $\frac{I_1}{\sigma}$

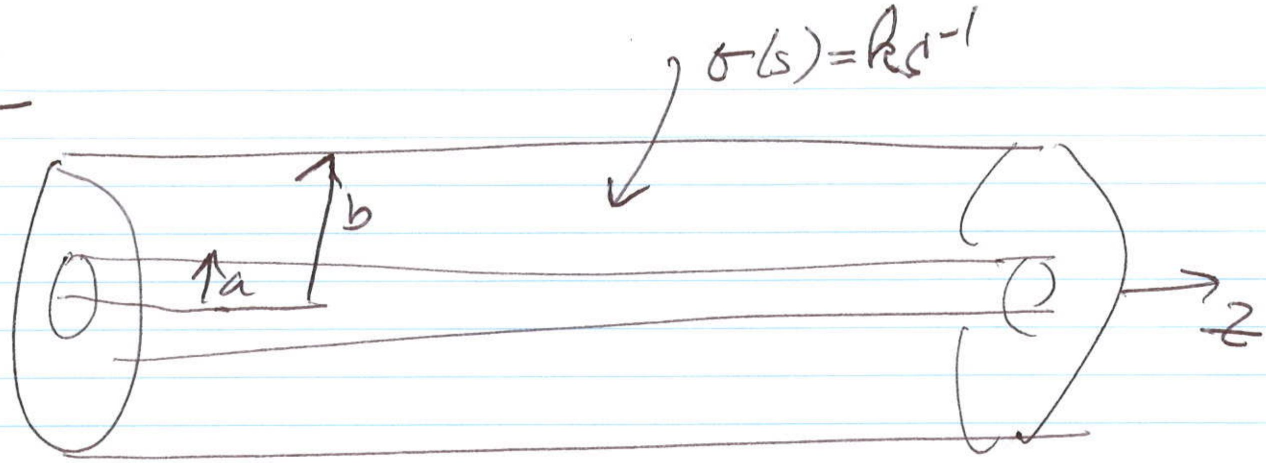
$$\Longrightarrow \boxed{R = \frac{\epsilon_0}{\sigma C}}$$

b) $CV = Q \rightarrow C\frac{dV}{dt} = \frac{dQ}{dt} = -I = -\frac{V}{R}$

$$\Longrightarrow C\frac{dV}{dt} = -\frac{V}{R} \longrightarrow \boxed{V = V_0 e^{-t/RC}}$$

note: $RC = \frac{\epsilon_0}{\sigma}$ from part a

2.4



a) find the resistance, R, between the cylinders.

$$\vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot\sigma\vec{E} = \vec{\nabla}\cdot(k\vec{E}/s) = 0$$

$$(k/s)\vec{\nabla}\cdot\vec{E} + (\vec{E}\cdot\vec{\nabla})(k/s) = 0$$

note: $\vec{E} = -\vec{\nabla}V$

$$\rightarrow -(k/s)\nabla^2 V + \left(-\frac{\partial V}{\partial s}\frac{\partial}{\partial s} - \cancel{\frac{1}{s}\frac{\partial V}{\partial \phi}\frac{1}{s}\frac{\partial}{\partial \phi}} - \cancel{\frac{\partial V}{\partial z}\frac{\partial}{\partial z}}\right)(k/s) = 0$$

$$= -(k/s)\nabla^2 V + \frac{\partial V}{\partial s}\frac{k}{s^2} = 0$$

$$-(k/s)\frac{1}{s}\frac{\partial}{\partial s}s\frac{\partial}{\partial s}V + \frac{\partial V}{\partial s}\frac{k}{s^2} = 0$$

$$-\frac{k}{s^2}\left[\cancel{\frac{\partial V}{\partial s}} + s\frac{\partial^2 V}{\partial s^2}\right] + \cancel{\frac{k}{s^2}\frac{\partial V}{\partial s}} = 0$$

$$\Rightarrow -\frac{k}{s}\frac{\partial^2 V}{\partial s^2} = 0$$

and $\boxed{V = a_0 + a_1 s}$