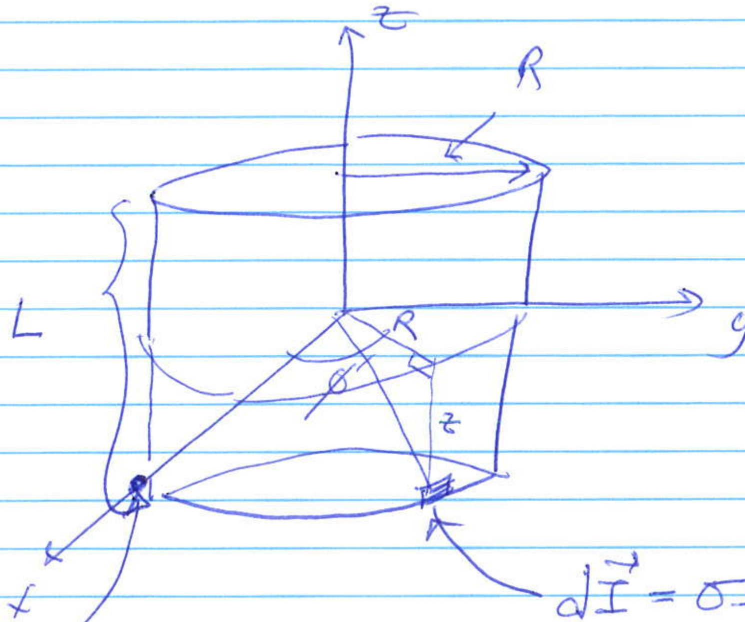
Prob 5.61



$$d\vec{I} = \sigma\Omega_0 R\hat{\phi} = \sigma\Omega_0 R\left(-\sin\phi', \cos\phi', 0\right)$$

$$\vec{r} = (s, 0, z) \qquad \vec{r}\,' = \left(R\cos\phi', R\sin\phi', z'\right)$$

$$d\vec{A} = \frac{\mu_0}{4\pi}\,\frac{\sigma\Omega_0 R\left(-\sin\phi', \cos\phi', 0\right) R\,d\phi'\,dz'}{\sqrt{(s-R\cos\phi')^2 + R^2\sin^2\phi' + (z-z')^2}}$$

By symmetry, the x component drops out

$$\Rightarrow dA_y = \frac{\mu_0\sigma\Omega_0 R^2}{4\pi}\left(\frac{\cos\phi'\,d\phi'\,dz'}{\sqrt{(s-R\cos\phi')^2 + R^2\sin^2\phi' + (z-z')^2}}\right)$$

hmmm

Use $s >> R$

$$dA_y \simeq \frac{\mu_0 \sigma \Omega_0 R^2}{4\pi}\left[\frac{\cos\phi' d\phi' dz'}{\sqrt{s^2\left(1-2\frac{R\cos\phi'}{s}\right)+R^2\sin^2\phi' + (z-z')^2}}\right]$$

$$\simeq \frac{\mu_0 \sigma \Omega_0 R^2}{4\pi}\left[\frac{\cos\phi' d\phi' dz'}{\sqrt{s^2+(z-z')^2 - 2sR\cos\phi'}}\right]$$

$$\simeq \frac{\mu_0 \sigma \Omega_0 R^2}{4\pi}\frac{\cos\phi' d\phi' dz'}{\sqrt{s^2+(z-z')^2}}\left[1+\frac{sR\cos\phi'}{s^2+(z-z')^2}\right]$$

Integrate over ϕ'

$$A_y \simeq \left(\frac{\mu_0 \sigma \Omega_0 R^3}{4}\right)\underbrace{\int_{-L/2}^{L/2}\frac{s\,dz'}{\left(s^2+[z-z']^2\right)^{3/2}}}_{\text{look up in tables}}$$

$$\simeq \frac{\mu_0 \sigma \Omega_0 R^3}{4}\, s\left[\frac{(z-z')}{s^2\sqrt{s^2+(z-z')^2}}\right]_{-L/2}^{L/2}$$

$$A_y = \frac{\mu_0 \sigma \Omega_0 R^3}{4}\frac{1}{s}\left[\frac{(z-L/2)}{\sqrt{s^2+(z-L/2)^2}} - \frac{(z+L/2)}{\sqrt{s^2+(z+L/2)^2}}\right]$$

Because of axial symmetry

$$\Rightarrow \vec{A} = \frac{\mu_0 \sigma \Omega_0 R^3}{4s}\left[\frac{(z-L/2)}{\sqrt{s^2+(z-L/2)^2}} - \frac{(z+L/2)}{\sqrt{s^2+(z+L/2)^2}}\right]\hat{\phi}$$

To get $\vec{B}$, do $\vec{\nabla}\times\vec{A}$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$= \cancel{\frac{\mu_0\sigma\Omega_0 R^3}{4}}\left\{ -\frac{\partial A_\phi}{\partial z}\hat{s} + 0\hat{\phi} + \frac{1}{s}\frac{\partial}{\partial s}\left(sA_\phi\right)\hat{z} \right\}$$

$$= \frac{\mu_0\sigma\Omega_0 R^3}{4}\left\{ -\frac{\hat{s}}{s}\left[\frac{1}{\sqrt{s^2+(z-\frac{L}{2})^2}} - \frac{(z-\frac{L}{2})^2}{\left(s^2+\left[z-\frac{L}{2}\right]^2\right)^{3/2}} - \frac{1}{\sqrt{s^2+(z+\frac{L}{2})^2}} + \frac{(z+\frac{L}{2})^2}{\left(s^2+\left[z+\frac{L}{2}\right]^2\right)^{3/2}} \right]\right.$$

$$\left. + \frac{\hat{z}}{s}\left[\frac{(z-\frac{L}{2})(-\frac{2s}{2})}{\left(s^2+\left[z-\frac{L}{2}\right]^2\right)^{3/2}} - \frac{(z+\frac{L}{2})(-\frac{2s}{2})}{\left(s^2+\left[z-\frac{L}{2}\right]^2\right)^{3/2}} \right]\right\}$$

now, a slightly questionable move; let "center of rod" mean the origin, not the axis.

Let $z=0$

$$\Rightarrow \vec{B} = \frac{\mu_0\sigma\Omega_0 R^3}{4}\left\{ 0\hat{s} + \frac{L}{\left(s^2+L^2/4\right)^{3/2}}\hat{z} \right\}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0\sigma\Omega_0 R^3 L}{4\left(s^2+L^2/4\right)^{3/2}}\hat{z}}$$