Homework 4

Due: February 8, 2013

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21. 6.1

22. 6.3

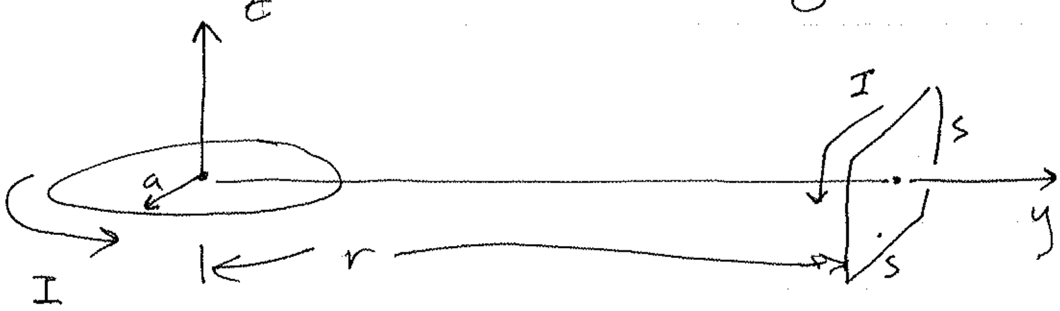
23. 6.5

24. 6.6

25. 6.25

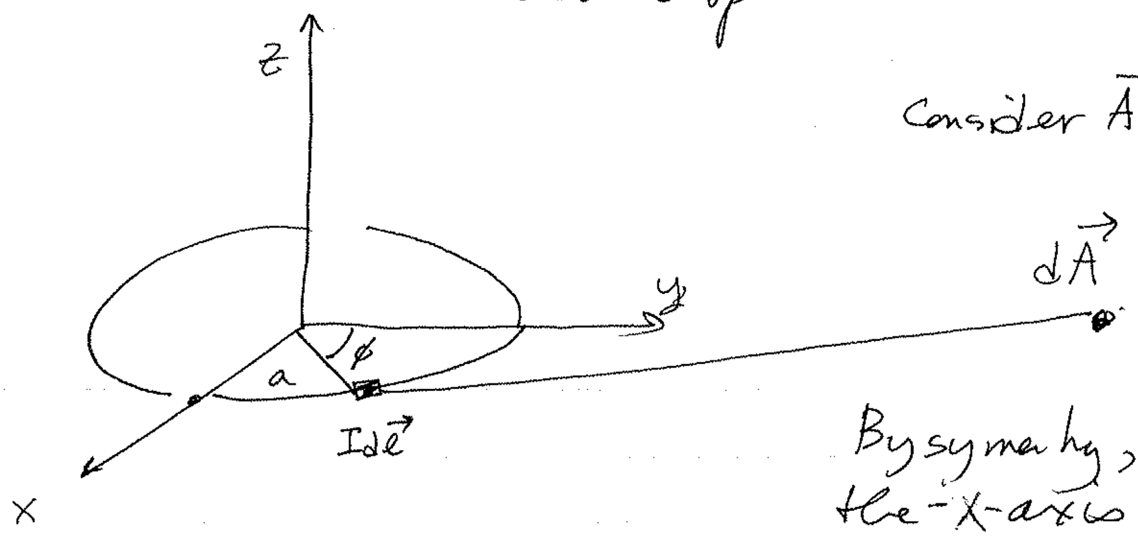
Prob 6.1

Q: What is the torque for the following configuration?



assume $r >> a, s$

for the square loop, $\vec{m} = I s^2 \hat{y}$. Now, what is $\vec{B}$ due to the circular loop?



Consider $\vec{A}$

By symmetry, $\vec{A}$ will be along the $-x$-axis and so,

$$d\vec{A} = \frac{-\mu_0}{4\pi}\left(\frac{I a d\phi \cos\phi}{\sqrt{a^2+r^2-2ar\cos\phi}}\right)\hat{x}$$

$$= \frac{-\mu_0 I a}{4\pi r}\left(\sum\left(\frac{a}{r}\right)^n P_n(\cos\phi)\right)\hat{x}\cos\phi\, d\phi$$

$$\vec{A} = -\frac{\mu_0 I a}{4\pi r} \int \sum \left(\frac{a}{r}\right)^n P_n(\cos\phi) \cos\phi \, d\phi \, \hat{x}$$

$$= -\frac{\mu_0 I a}{4\pi r} \int \hat{x} \left[\frac{a}{r} \cos^2\phi \, d\phi + \frac{a^2}{r^2} \frac{1}{2} (3\cos^2\phi - 1) \cos\phi \, d\phi + \cdots \right]$$

$$\simeq -\frac{\mu_0 I a}{4\pi r} \hat{x} \left[\frac{\pi a}{r} + 0 + O\left(\frac{1}{r^3}\right) \right] \leftarrow$$ By symmetry, argue that $\vec{A}$ is in the $\hat{\phi}$-direction

$$\vec{A} = \frac{\mu_0 I a^2}{4 r^2} \hat{\phi}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A} = \hat{r} \left[-\frac{\partial A_\phi}{\partial z} \right] + \hat{z} \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\phi) \right]$$

$$= -\hat{r} \left[0 \right] + \frac{\hat{z}}{r} \left[-\frac{\mu_0 I a^2}{4 r^2} \right]$$

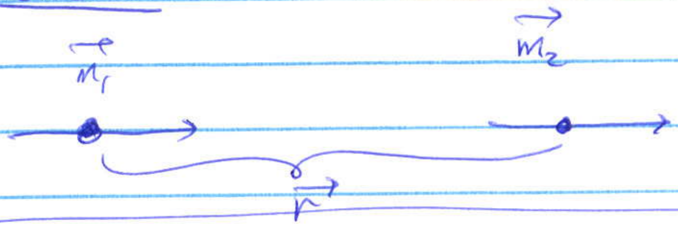
$$\boxed{\vec{B} = -\frac{\mu_0 I a^2}{4 r^3} \hat{z}}$$

The torque is then

$$\vec{N} = \vec{m} \times \vec{B}$$

$$= \frac{\mu_0}{4} \frac{I^2 a^2 s^2}{r^3}$$ about the x-axis. The loop rotates so that the $\vec{m}$ points downward (in the $-z$-dir^n)

Prob 6.3



$\boxed{\text{find the force on } \vec{m}_2 \text{ due to } \vec{m}_1}$

(i) Use $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$

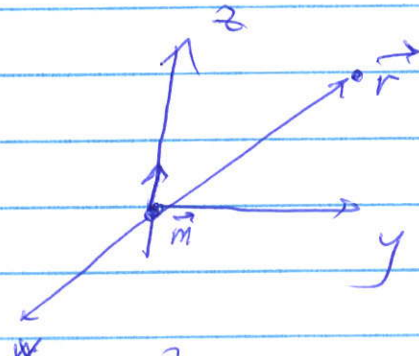
and/or

$$\vec{F} = \oint I d\vec{l} \times \vec{B}$$

(ii) The magnetic field of a dipole is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$$

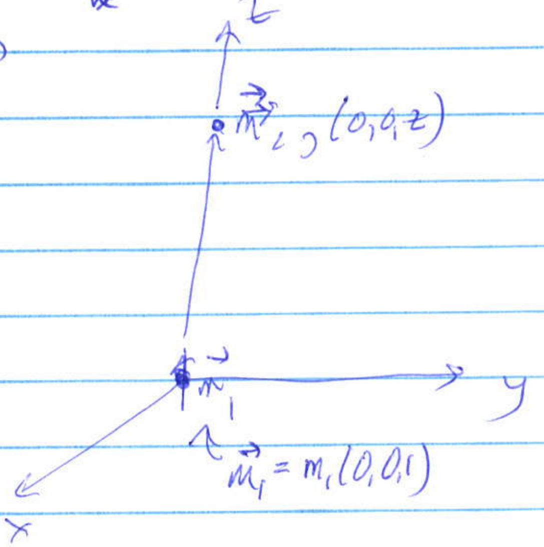
where



field of $\vec{m}$ at position $\vec{r}$

so,

①



$$\vec{B}_1 = \frac{\mu_0}{4\pi r^3}\left[3(m_1)\hat{z} - m_1(0,0,1)\right]$$

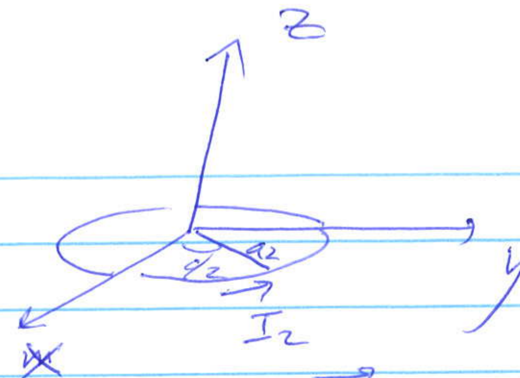
$$\Rightarrow \vec{m}_2\cdot\vec{B}_1 = \frac{\mu_0}{4\pi r^3}\left[3m_1m_2 - m_1m_2\right]$$

$$\Rightarrow \vec{F} = \vec{\nabla}(\vec{m}_2\cdot\vec{B}_1)$$

$$\vec{F} = -\frac{3\mu_0 m_1 m_2}{2\pi z^4}\hat{z}$$

② $\vec{F} = \oint I_2 \, d\vec{l} \times \vec{B}_1$

Use dipole field



$\Longrightarrow \vec{m}_2 = I_2 \pi a_2^2 \hat{z}$

$$\Rightarrow \vec{F} = \oint I_2 a_2 d\vec{\phi}_2 \times \frac{\mu_0 m_1}{4\pi}\left(\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right)$$

$$= \frac{\mu_0 m_1}{4\pi} a_2 I_2 \frac{1}{r^3} \oint d\vec{\phi}_2 \times \left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$= \frac{\mu_0 m_1 a_2 I_2}{4\pi r^3} \oint \left[d\phi_2 \left(2\cos\theta\,\hat{\theta} - \sin\theta\,\hat{r}\right)\right]$$

$$\cancel{= \frac{\mu_0 m_1 a_2 I_2}{2 r^3}\left[2\cos\theta\,\hat{\theta} - \sin\theta\,\hat{r}\right]}$$

resolve into $\hat{s}$ & $\hat{z}$ components; $\hat{s}$ components integrate to 0 and so,

$$F_z = \frac{\mu_0 m_1 a_2 I_2}{4\pi r^3}\left[-2\pi \times 2\cos\theta\sin\theta\,\hat{z} - \sin\theta\cos\theta\, 2\pi\,\hat{z}\right]$$

$$= -\frac{3}{2}\left(\frac{\mu_0 m_1 a_2 I_2}{r^3}\right)\cos\theta\sin\theta\,\hat{z}$$

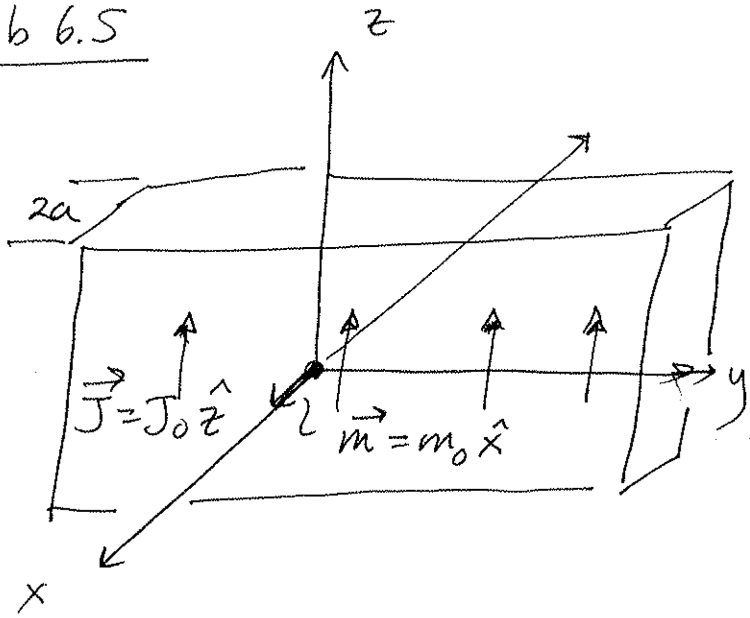
note: $a_2 = r\sin\theta$

$\theta \to 0$ (small loop) $\Rightarrow \cos\theta \to 1$

$$= -\frac{3}{2}\frac{\mu_0 m_1 I_2 a_2^2}{r^4}\cos\theta\,\hat{z}$$

$$\Rightarrow F_z \simeq -\frac{3}{2\pi}\frac{\mu_0 m_1 m_2}{r^4}\hat{z}, \quad \boxed{m_2 = I_2 \pi a_2^2}$$

Prob 6.5



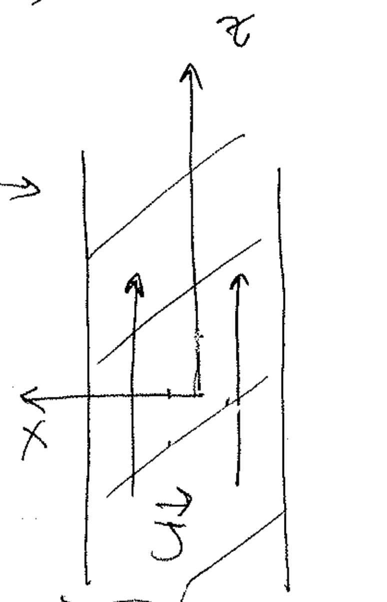
a) Q: Find the force on the dipole using

(i) $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$ (6.3)

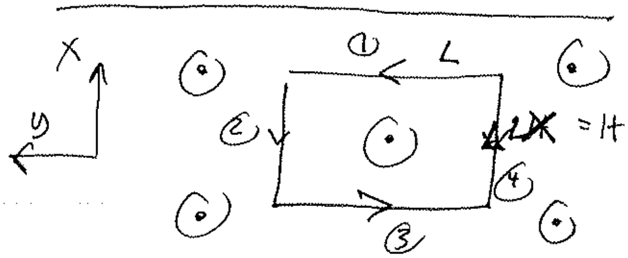
(ii) $\vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B}$ (6.5)

What is $\vec{B}$ for this slab?

- By symmetry, field will be in $\pm y$-direction.
- To find field, use Ampere's law. look at view from $+z$ direction:



any x-component cancels and z is // to $\vec{J}$.



$$\oint \vec{B}\cdot d\vec{l} = \cancel{B_x^2 H}^{\,0} + B_y^3(-x)L + \cancel{B_x^4 H}^{\,0} + B_y^1(x)L$$

$$= \mu_0 J L 2x$$

$$B_y(x) - B_y(-x) = 2\mu_0 J_0 x$$

at $x=0 \longrightarrow B_y(+0) = B_y(-0) = 0$ by symmetry

The sol^{n} is then,

$$B_y(x)\hat{y} = \begin{cases} \mu_0 J_0 x \hat{y} & x>0 \\ \mu_0 J_0 x \hat{y} & x<0 \end{cases}$$

in the slab.

The force on the dipole is thus,

$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

$$= \vec{\nabla}\big(\underbrace{\cancel{m_0 \hat{x} \cdot B_y(x)\hat{y}}}_{\to 0}\big)$$

$$= 0$$

b) Do the same for a dipole, $\vec{m} = m_0 \hat{y}$

$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B}) \leftarrow m_0 \mu_0 J_0 x$$

$$= \vec{\nabla}(m_0 \hat{y} \cdot B_y(x)\hat{y})$$

$$= \hat{x}\frac{\partial}{\partial x}(\mu_0 m_0 J_0 x)$$

$$\boxed{\vec{F} = \mu_0 m_0 J_0 \hat{x}}$$

c) $\vec{F} = \vec{\nabla}(\vec{p}\cdot\vec{E})$ $\quad 0,\ \vec{\nabla}\times\vec{E} = 0$ in electrostatics $\quad \vec{p}$ is constant

$= \vec{p}\times\cancel{(\vec{\nabla}\times\vec{E})} + \vec{E}\times(\vec{\nabla}\times\vec{p}) + (\vec{p}\cdot\vec{\nabla})\vec{E}$

$+ (\vec{E}\cdot\vec{\nabla})\vec{p}$ $\quad$ (ID #4)

$\vec{p}$ is constant

$$\boxed{\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}}$$

In magnetostatics, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

d) Use $\vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B}$ and redo forces (part b)

(i) $\vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B}$

$= m_0 \frac{\partial}{\partial x}\left[B_y(x)\hat{y}\right]$

$= m_0\mu_0 J_0 \hat{y}$

(ii) $\vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B}$

$= m_0 \frac{\partial}{\partial y}\left(B_y(x)\hat{y}\right)$

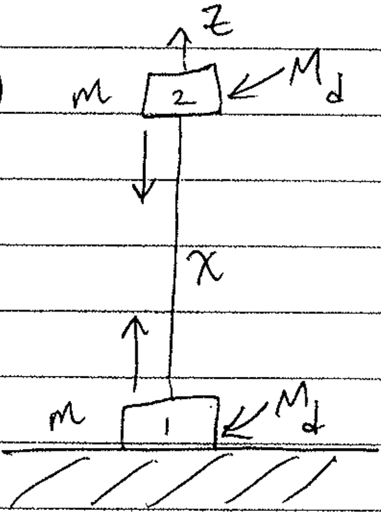
$= 0$

28. Prob 6.6

Material	N of e^-'s	Para-	Dia-
Al	13, $3s^2 3p$	✓	
Cu	29, $3d^{10}4s$	✓	
$CuCl_2$	63	✓	
C	6, $2s^2 2p^2$		✓
Pb	82		✓
N_2	14		✓
NaCl	28		✓
Na	11, 3s	✓	
S	16, $3s^2 3p^4$		✓
H_2O	10		✓

susceptibilities, χ_m

Prob 6.25

a) 

$$\vec{B}_1 = \hat{z}\,\frac{\mu_0}{4\pi}\frac{m}{z^3}\left(2\hat{z}\right)$$

$$F_z = \frac{\partial}{\partial z}\left(-m\,\frac{\mu_0}{4\pi}\frac{m\,2}{z^3}\right)$$

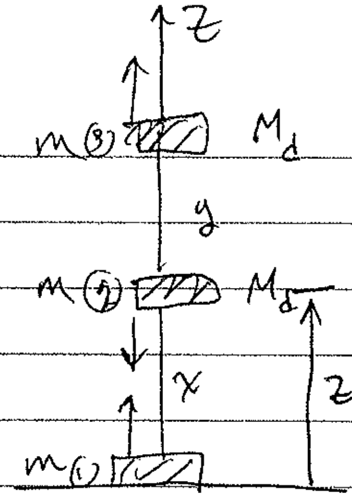
$$= \frac{3\mu_0}{2\pi}m^2\frac{1}{z^4}$$

In equilibrium,

$$0 = \frac{3\mu_0 m^2}{2\pi}\frac{1}{z_e^4} - M_d\, g$$

$$\longrightarrow \boxed{z_e = \left(\frac{3\mu_0 m^2}{2\pi M_d g}\right)^{1/4}}$$

b)



②: $\vec{B} = \frac{\mu_0 m}{4\pi}\left(\frac{2\hat{z}}{x^3}\right) + \frac{\mu_0 m}{4\pi}\left(\frac{2\hat{z}}{y^3}\right)$

$x \to z \qquad y \to (x+y-z)$

$$F_z = \frac{\mu_0 m^2}{2\pi}\left[\frac{1}{x^4} + \frac{-1}{(x+y-z)^4}\right]3$$

$$\rightarrow \boxed{0 = -M_d g + \frac{\mu_0 m^2}{2\pi}3\left[\frac{1}{x^4} - \frac{1}{y^4}\right]} \quad (A)$$

③: $\vec{B} = \frac{\mu_0 m \hat{z}}{2\pi (z+y)^3} + \frac{\mu_0 m \hat{z}}{2\pi (x+y-z)^3}$

$$F_z = \frac{3\mu_0 m^2}{2\pi}\left[\frac{1}{(z+y)^4} + \frac{-1}{(x+y-z)^4}\right]$$

$$\rightarrow \boxed{0 = -M_d g - \frac{3\mu_0 m^2}{2\pi}\left[\frac{1}{(x+y)^4} - \frac{1}{y^4}\right]} \quad (B)$$

Subtract (B) from (A)

$$\frac{3\mu_0 m^3}{2\pi}\left[\frac{1}{x^4} - \frac{1}{y^4}\right] + \frac{3\mu_0 m^2}{2\pi}\left[-\frac{1}{y^4} + \frac{1}{(x+y)^4}\right] = 0$$

factor out $\frac{3 \gamma_0 m^2}{2\pi}$ and multiply by y^4

$$0 = \frac{y^4}{x^4} - 1 - 1 + \frac{y^4/x^4}{(1+y/x)^4}$$

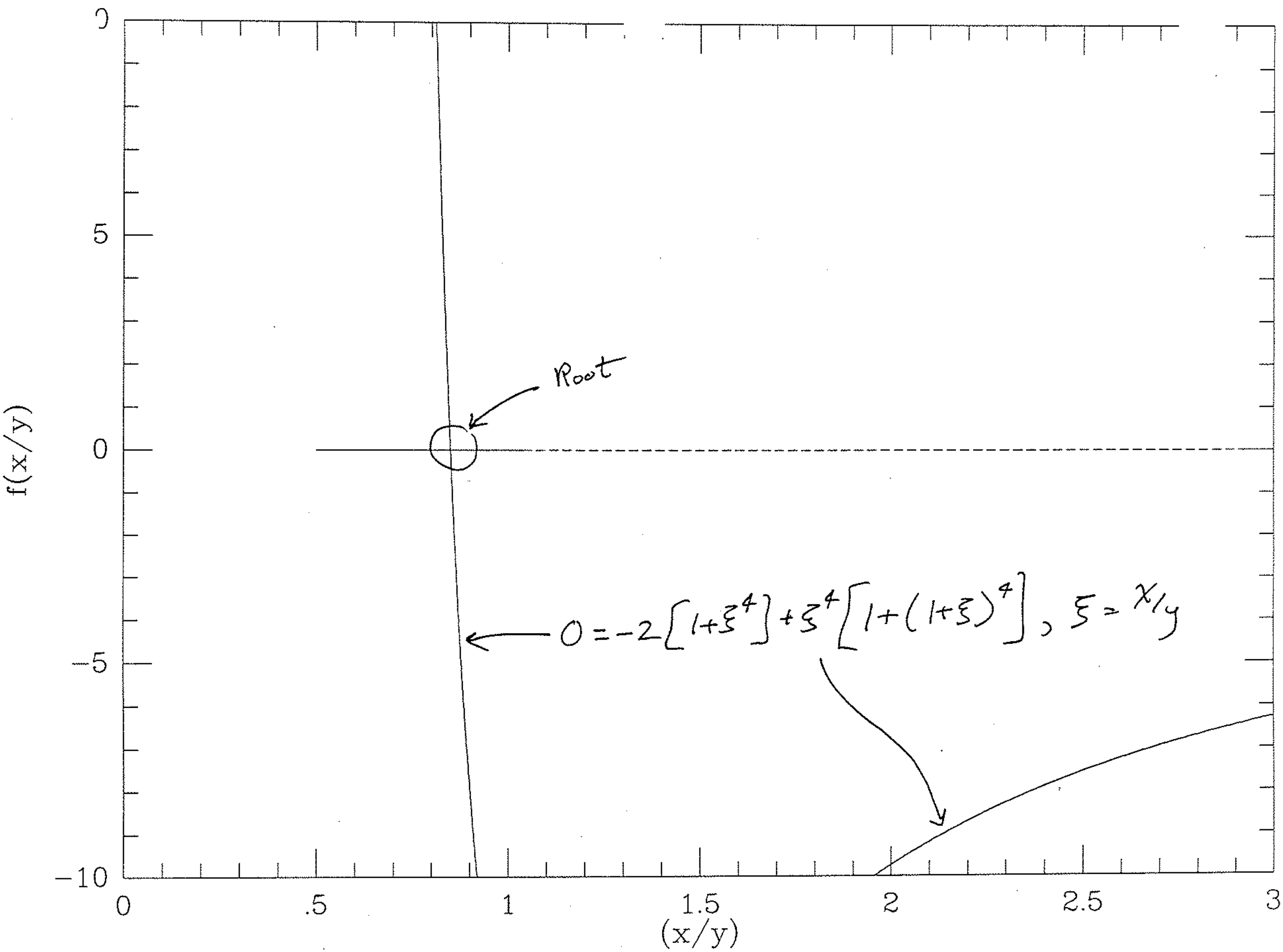
Let $\xi = \frac{y}{x}$

$$0 = -2 + \xi^4 + \frac{\xi^4}{(1+\xi)^4}$$

$$\rightarrow \boxed{0 = -2(1+\xi)^4 + (1+\xi)^4 \xi^4 + \xi^4}$$

By plotting, I find that

$$\xi \approx 1.176 \longrightarrow \boxed{\frac{x}{y} \approx 0.8503 \pm 0.0002}$$



f(x/y)
0
5
0
−5
−10
0
.5
1
1.5
2
2.5
3
(x/y)
Root
$0 = -2\left[1+\xi^4\right] + \xi^4\left[1+(1+\xi)^4\right]$, $\xi = x/y$