NAME Key

PHYSICS 413

Magnetostatics

Mid-term Examination

May 7, 2003

VECTOR DERIVATIVES

Cartesian. $dl = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $dl = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $dl = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$.

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :
$$\begin{cases} \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \end{cases}$$

In matter :
$$\begin{cases} \nabla\cdot\mathbf{D} = \rho_f \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :
$$\begin{cases} \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :
$$\begin{cases} \mathbf{P} = \epsilon_0\chi_e\mathbf{E}, \quad \mathbf{D} = \epsilon\mathbf{E} \\ \mathbf{M} = \chi_m\mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials
$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

Lorentz force law
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

Energy, Momentum, and Power

Energy : $$U = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$$

Momentum : $$\mathbf{P} = \epsilon_0\int(\mathbf{E}\times\mathbf{B})\,d\tau$$

Poynting vector : $$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

Larmor formula : $$P = \frac{\mu_0}{6\pi c}q^2a^2$$

FUNDAMENTAL CONSTANTS

ϵ_0	=	$8.85\times10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	=	$4\pi\times10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	=	$3.00\times10^{8}\,\mathrm{m/s}$	(speed of light)
e	=	$1.60\times10^{-19}\,\mathrm{C}$	(charge of the electron)
m	=	$9.11\times10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Question 1:

A charged annulus with inner radius a and outer radius b spins with frequency Ω about its center. The annulus has uniform surface charge density σ_o.

a. Find the magnetic dipole moment $\mathbf{m}$ of the annulus.

b. What is the vector potential $\mathbf{A}$ and magnetic field $\mathbf{B}$ for $r \gg b$, where r is the distance to the field point?

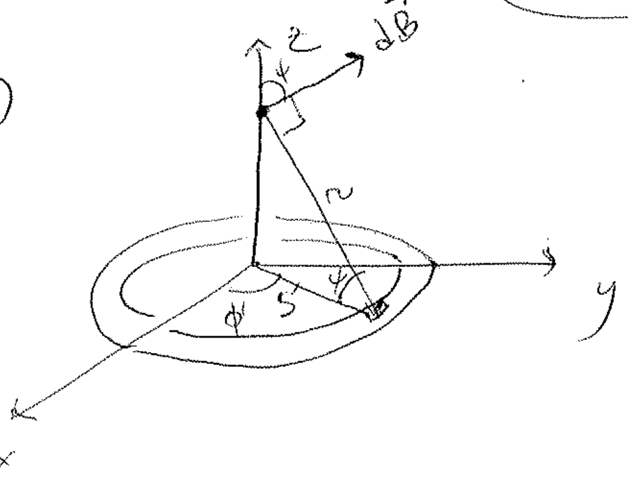
c. Find the $\mathbf{B}$ along the z-axis, the symmetry axis of the annulus, valid for all z. Find the radial component of $\mathbf{B}$ for points just off of the z-axis.

a) $$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{K}\,dS = \frac{1}{2}\int s\,\overbrace{\sigma_0\Omega s}^{K}\,\overbrace{(s\,d\phi\,ds)}^{dS}\,\hat{z} = \pi\sigma_0\Omega\int s^3\,ds\,\hat{z}$$

$$\boxed{\vec{m} = \frac{\pi\sigma_0\Omega}{4}\left(b^4-a^4\right)\hat{z}}$$

b) $$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2},\quad \vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$

c)



$$dB_z = \frac{\mu_0}{4\pi}\frac{\sigma_0\Omega s'}{\mathscr{r}^2}\cos\psi\,(s'd\phi'ds')$$

$$= \frac{\mu_0\sigma_0\Omega}{4\pi}\frac{s'^3}{\mathscr{r}^3}\,ds'd\phi'$$

$$dB_z = \frac{\mu_0\sigma_0\Omega}{2}\frac{s'^3\,ds'}{(z^2+s'^2)^{3/2}}$$

$$\rightarrow B_z = \frac{\mu_0\sigma_0\Omega}{2}\int\frac{s'^3\,ds'}{(s'^2+z^2)^{3/2}}$$

Let: $w = s'^2+z^2 \longrightarrow dw = 2s'ds' \quad s'^2 = w - z^2$

$$\rightarrow B_z = \frac{\mu_0\sigma_0\Omega}{2}\int\frac{\frac{1}{2}dw}{w^{3/2}}\left(w-z^2\right)$$

$$B_z = \frac{\mu_0 \sigma_0 \Omega}{4} \int \left(w^{-1/2} - \frac{z^2}{w^{3/2}} \right) dw$$

$$= \frac{\mu_0 \sigma_0 \Omega}{4} \left[2w^{1/2} + 2\frac{z^2}{w^{1/2}} \right]_{a^2+z^2}^{b^2+z^2}$$

$$\boxed{\vec{B} = \frac{\mu_0 \sigma_0 \Omega}{2} \hat{z} \left[\sqrt{b^2+z^2} - \sqrt{a^2+z^2} + \frac{z^2}{\sqrt{b^2+z^2}} - \frac{z^2}{\sqrt{a^2+z^2}} \right]}$$

<u>find B_s near $s=0$</u>

$$\vec{\nabla} \cdot \vec{B} = 0 \longrightarrow \frac{1}{s}\frac{\partial sB_s}{\partial s} + \cancelto{0}{\frac{1}{s}\frac{\partial B_\phi}{\partial \phi}} + \frac{\partial B_z}{\partial z} = 0$$

$$\longrightarrow sB_s \approx -\frac{s^2}{2}\left(\frac{\partial B_z}{\partial z}\right)\Bigg|_{s=0}$$

$$\boxed{B_s \approx -\frac{s}{2}\left(\frac{\partial B_z}{\partial z}\right)\Bigg|_{s=0}}$$

Question 2:

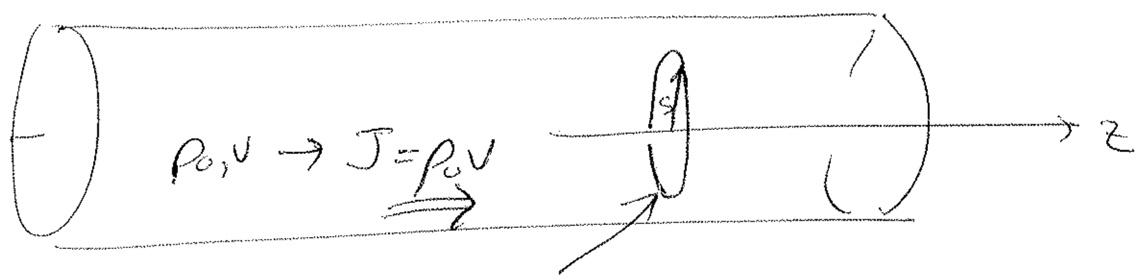
A long, cylindrical beam of protons moves down an evacuated pipe with speed v. The beam has cylindrical radius R and the protons are distributed uniformly in the beam with charge density $\rho_\circ$.

a. Find the magnetic field **B**.

b. Find the Lorentz force per unit length felt by the beam. Does the Lorentz force act to focus or defocus the beam?

c. Find the equation-of-motion for the beam including both the electric force and the Lorentz force. For what speed v do the electric and magnetic forces cancel each other?

a)



Amperian loop

$$\oint \vec{B}\cdot d\vec{l} = \mu_\circ \int \vec{J}\cdot d\vec{S}$$

$$B_\phi 2\pi s = \mu_\circ \rho_0 v \pi s^2 \longrightarrow \boxed{\vec{B} = \frac{\mu_\circ \rho_0 v}{2} s\,\hat{\phi}}$$

$s < R$

$$\boxed{\vec{B} = \frac{\mu_\circ \rho_0 v R^2}{2s}}$$

$s > R$

b) $$d\vec{F} = (\vec{J}\,d^3x)\times\vec{B}$$

inward

$$= \rho_0 v \underbrace{d^3x}_{s\,d\phi\,ds\,dz}\left(\frac{\mu_0\rho_0 v}{2}s\right)(-\hat{s})$$

$$\Longrightarrow \frac{d\vec{F}}{dz} = \rho_0 v\left(\frac{\mu_\circ}{2}\rho_0 v\right)(-s\hat{s})\,s\,ds\,d\phi$$

$$\frac{d\vec{F}}{dz} = (\rho_0 v)^2 \mu_0 \pi \left(-\int s^2 ds \right) \hat{s}$$

$$= -\frac{\mu_0 \pi}{3} (\rho_0 v)^2 \left[R^3 - 0 \right] \hat{s}$$

$$\boxed{\frac{d\vec{F}}{dz} = \frac{\mu_0 \pi R^3}{3} (\rho_0 v)^2 \hat{s}}$$

$\Longrightarrow$ focuses the beam

Question 3:

A circular wire loop of radius a and current I_a, is placed at the midpoint of the axis of a solenoid of radius R and length L. The axis for the wire loop makes at angle ϕ with the axis of the solenoid. The solenoid coil carries current I_s and is wound with N coils per unit length.

a. Find the field on the axis of the solenoid. Ignore the contribution of the wire loop to the field.

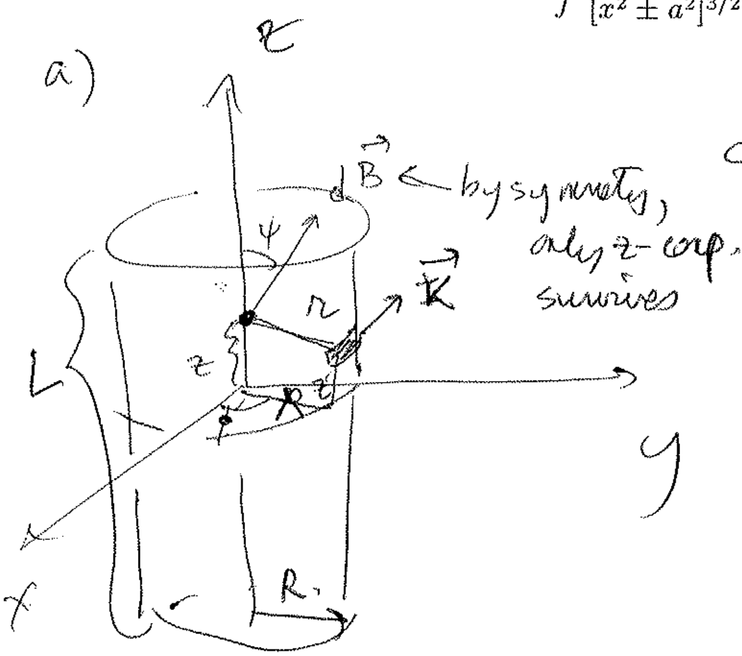
b. If the location of the center of the loop is fixed at the midpoint of the solenoid (but is free to rotate), what is the torque on the wire loop? Draw a sketch indicating how the wire loop moves and its orientation when it is in a stable equilibrium. Include $\mathbf{B}$ and the coordinate system on your sketch. Let $L \to \infty$, that is, consider an infinite solenoid. The wire loop does not have angular momentum.

c. If the loop is rotated to $\phi = 0°$ and tranlsated to $z = z_o > 0$ and then released, does it return to the midpoint of the solenoid or does it move away from the midpoint of the solenoid? For this part, use the $\mathbf{B}$−field for the finite length solenoid.

note:

$$\int \frac{dx}{[x^2 \pm a^2]^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}} \qquad (1)$$

a)



$$dB_z = \frac{\mu_0}{4\pi}\frac{(NI_a)}{r^2}\cos\psi\,[R\,d\phi\,dz]$$

$$\cos\psi = \frac{R}{r}$$

$$= \frac{\mu_0 N I_a R^2}{4\pi}\left(\frac{d\phi\,dz}{r^3}\right)$$

$$= \frac{\mu_0 N I_a R^2}{2}\left(\frac{dz'}{[R^2+(z-z')^2]^{3/2}}\right)$$

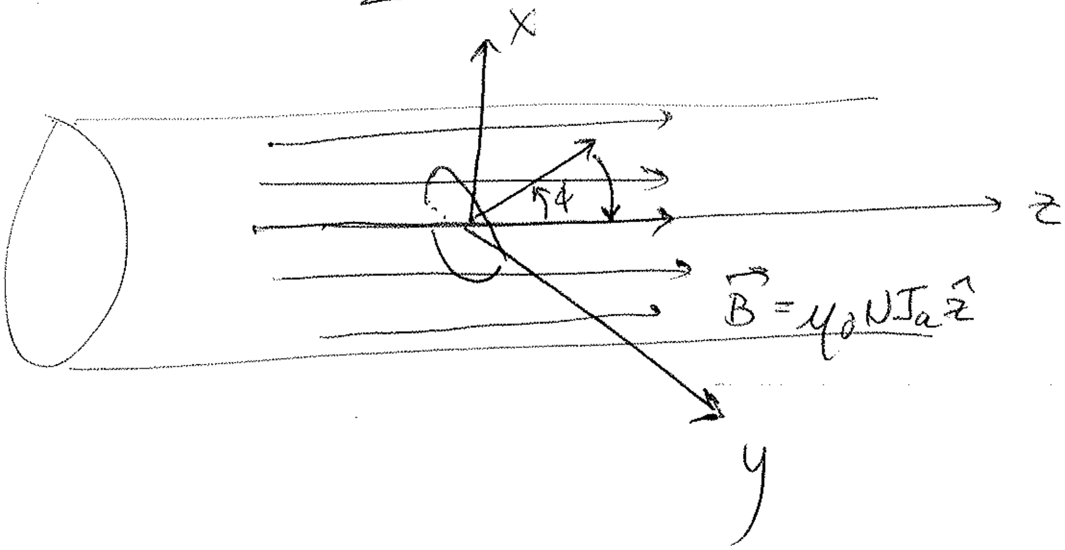
$$\to B_z = \frac{\mu_0 N I_a R^2}{2}\left[\frac{-(z-z')}{R^2\sqrt{R^2+(z-z')^2}}\right]_{-L/2}^{L/2}$$

ad $\boxed{\vec{B}_z = \frac{\mu_0 N I_a \cancel{R^2}}{2\cancel{R^2}}\left[\frac{(L/2-z)}{\sqrt{R^2+(L/2-z)^2}} + \frac{(L/2+z)}{\sqrt{R^2+(L/2+z)^2}}\right]\hat{z}}$

b) let $L \longrightarrow \infty$ for $z=0$

$$\vec{B}_z = \frac{\mu_0 N I_a}{2}\left[\frac{L}{\sqrt{R^2+L^2/4}}\right]\hat{z} \quad , z=0$$

$$\xrightarrow[L\to\infty]{} \frac{\mu_0 N I_a}{2}\, 2\,\hat{z} = \mu_0 N I_a \hat{z} \;\checkmark$$



$$\vec{N} = \vec{m} \times \vec{B} \text{ , where } |\vec{m}| = I_a \pi a^2 \text{ \& } \vec{B} = \mu_0 N I_a \hat{z}$$

$$\boxed{\vec{N} = I_a \pi a^2 \mu_0 N I_a \sin\phi \,(-\hat{y})}$$

c) Find $\vec{F}$

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) = \hat{z}\frac{\partial}{\partial z}\left(\overbrace{\frac{\mu_0 N I_a}{2} I_a \pi a^2}^{\text{constant}}\left\{\frac{\frac{L}{2}-z}{\sqrt{R^2+(\frac{L}{2}-z)^2}} + \frac{\frac{L}{2}+z}{\sqrt{R^2+(\frac{L}{2}+z)^2}}\right\}\right)$$

$\phi = 0°$

$$\propto \hat{z}\left\{\frac{\frac{L}{2}(\frac{L}{2}-z)}{\left(R^2+(\frac{L}{2}-z)^2\right)^{3/2}} - \frac{1}{\sqrt{R^2+(\frac{L}{2}-z)^2}} - \frac{\frac{L}{2}(\frac{L}{2}+z)}{\left(R^2+(\frac{L}{2}+z)^2\right)^{3/2}} + \frac{1}{\sqrt{R^2+(\frac{L}{2}+z)^2}}\right\}$$