Homework 5

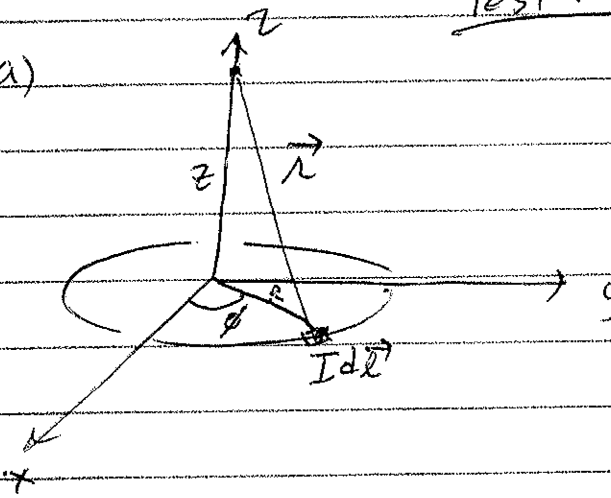
Due: February 13, 2013

Re-do of Test 1.

Copy of Test 1

Test 1

a)



for 1 loop

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I dl(-\sin\phi, \cos\phi, 0) \times (-R\cos\phi, -R\sin\phi, z)}{(R^2+z^2)^{3/2}}$$

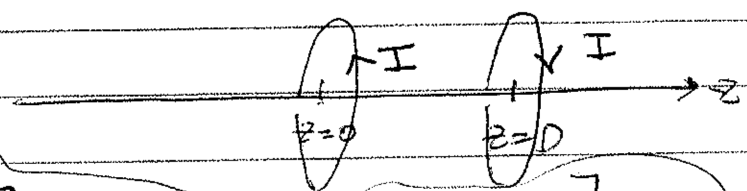
$$= \frac{\mu_0 I dl}{4\pi} \frac{(z\cos\phi, +z\sin\phi, R\sin^2\phi + R\cos^2\phi)}{(R^2+z^2)^{3/2}}$$

dB_x, dB_y will integrate to 0 over ϕ

$$\rightarrow B_z = \frac{\mu_0 I\, 2\pi R}{4\pi} \frac{R}{(R^2+z^2)^{3/2}}$$

$$= \frac{\mu_0}{2} \frac{IR^2}{(R^2+z^2)^{3/2}}$$

for 2 loops



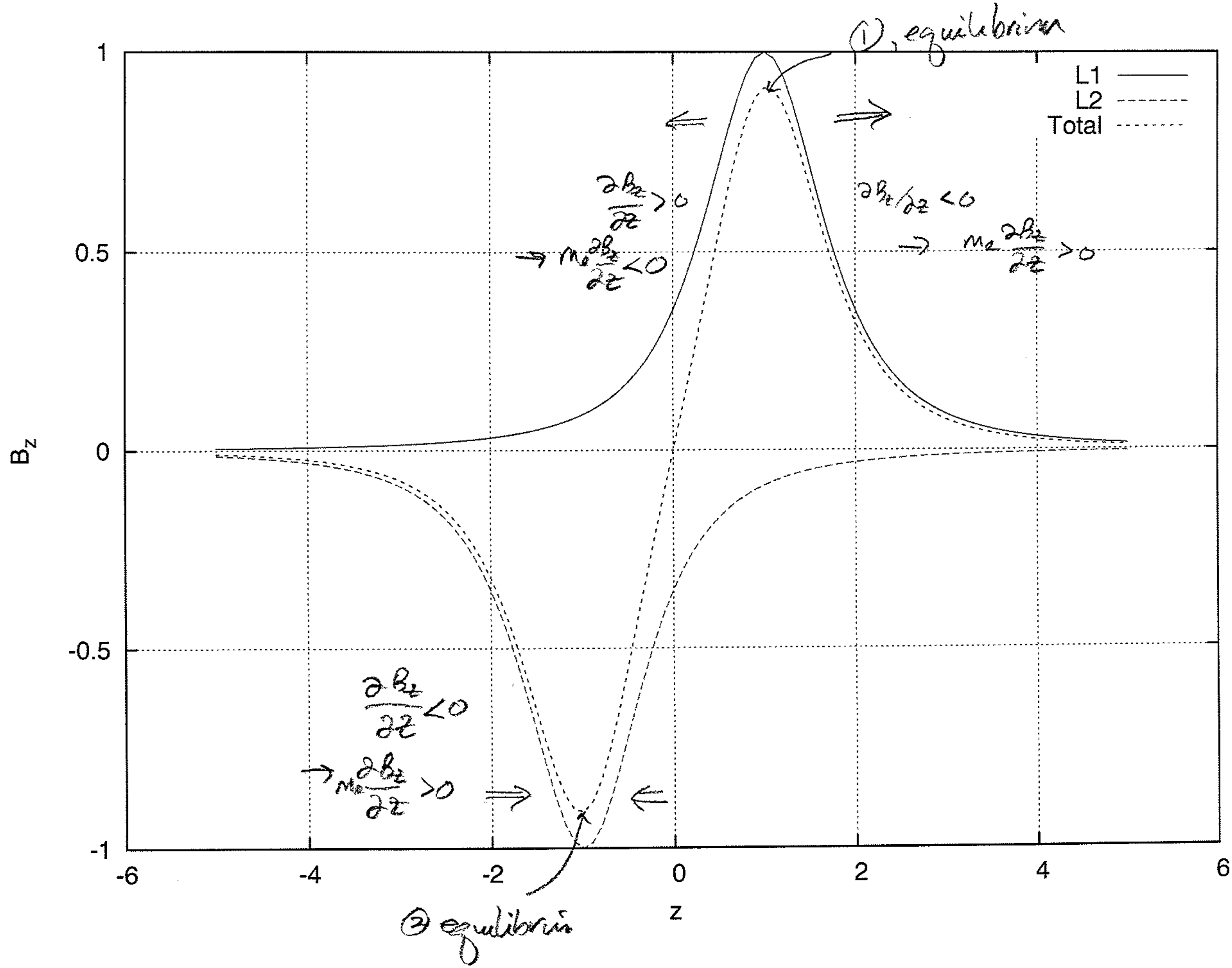
$$\boxed{\vec{B}_{z,total} = \frac{\mu_0 I R^2}{2}\left[\frac{1}{(R^2+[z-D]^2)^{3/2}} - \frac{1}{(R^2+z^2)^{3/2}}\right]\hat{z}}$$

b) Find on-axis $\vec{B}$ for $r \gg R$ and $r \gg D$

$$\vec{B}_{z,total} = \frac{\mu_0 I R^2}{2z^3}\left[\frac{1}{\left(\frac{R^2}{z^2}+\left[1-\frac{D}{z}\right]^2\right)^{3/2}} - \frac{1}{\left(1+\frac{R^2}{z^2}\right)^{3/2}}\right]\hat{z}$$

$$\simeq \frac{\mu_0 I R^2}{2z^3}\left[\cancel{1} - \frac{3}{2}\left(-2\frac{D}{z} + \cancel{\frac{R^2}{z^2}}\right) - \cancel{1} - \left(-\frac{3}{2}\right)\left(\cancel{\frac{R^2}{z^2}}\right)\right]\hat{z}$$

$$\simeq \frac{\mu_0 I R^2}{2z^3}\left[3\frac{D}{z}\right]\hat{z} \longrightarrow \boxed{\vec{B}_{z,total} \simeq \frac{3}{2}\frac{\mu_0 I R^2}{z^3}\left(\frac{D}{z}\right)\hat{z}}$$



① equilibrium
② equilibrium
L1
L2
Total
$\frac{\partial B_z}{\partial z} > 0$
$\rightarrow m_z \frac{\partial B_z}{\partial z} < 0$
$\partial B_z / \partial z < 0$
$\rightarrow m_z \frac{\partial B_z}{\partial z} > 0$
$\frac{\partial B_z}{\partial z} < 0$
$\rightarrow m_z \frac{\partial B_z}{\partial z} > 0$
B_z
z

Arbitrary Units for B_z and $R/D = 1/2$.

a) for one e^- w/ $\vec{S}$ along z-axis,

$$\vec{m}_e = \frac{g_e}{2}\left(\frac{e}{2m}\right)\vec{S} \Rightarrow \vec{m}_e \text{ along z-axis,}$$

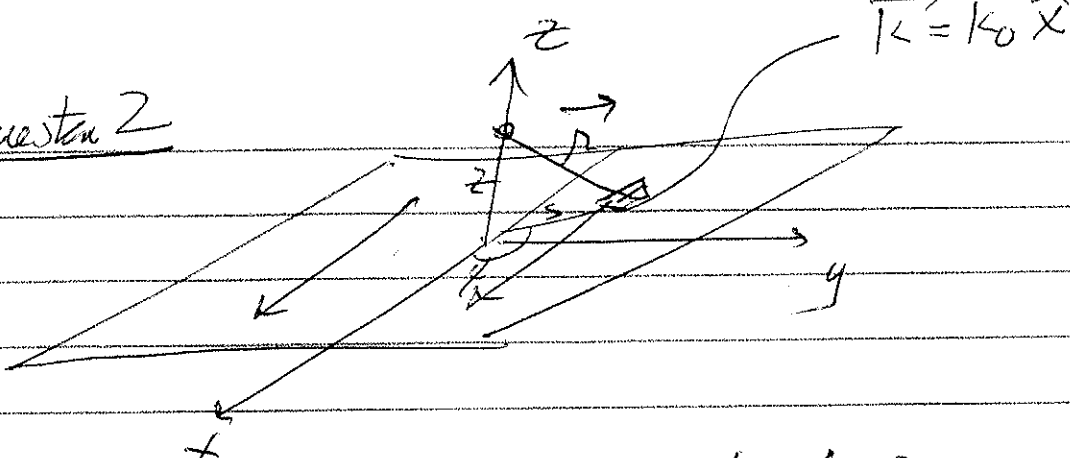
but in $-\hat{z}$ direction

b) equilibria where $\frac{\partial B_z}{\partial z} = 0 \Rightarrow$ at ① & ②

① is unstable; force is destabilizing $\vec{F} = \vec{\nabla}(\vec{m}_e \cdot \vec{B})$
pushes e^- away

② is stable, force is restoring $\vec{F} = \vec{\nabla}(\vec{m}_e \cdot \vec{B})$
pulls back

Question 2



$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{(K_0,0,0)\times(-x',-y',z)\,dS}{\left[x'^2+y'^2+(z)^2\right]^{3/2}}$$

$$= \frac{\mu_0 K}{4\pi}\frac{(0,-K_0 z,-K_0 y')\,dS}{\left[x'^2+y'^2+z^2\right]^{3/2}}$$

Because we can look at point $-y'$, we see that the z-component of $d\vec{B}$ cancels by symmetry $\Rightarrow$ we only need consider B_y

$$B_y = \frac{\mu_0 K_0}{4\pi}\int_0^{2\pi}\int_0^{\infty}\frac{-\cancel{K_0} z\, s\, d\phi\, ds}{(s^2+z^2)^{3/2}} \quad ; \text{where } s^2 = x'^2+y'^2$$

$$= -\frac{\mu_0 K_0 \cancel{2\pi} z}{\cancel{4\pi}2}\frac{1}{\cancel{2}}\left[\frac{-\cancel{2}}{(s^2+z^2)^{1/2}}\right]_0^{\infty}$$

$$= -\frac{\mu_0 K_0 z}{2}\left[-0-\left(-\frac{1}{|z|}\right)\right]$$

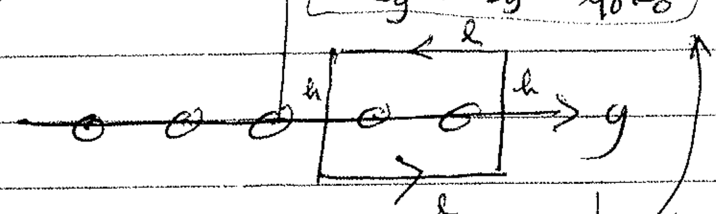
$$\boxed{B_y = -\frac{\mu_0 K_0}{2}\frac{z}{|z|}\hat{y}}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\int \vec{J}\cdot d\vec{S}$$

$$B_y^{>}\Big|_l^0 + B_y^{<}\Big|_0^l = \mu_0 K l$$

$$-B_y^{>} + B_y^{<} = \mu_0 K_0$$

or Use Ampere's law



independent of h

$$\Longrightarrow \boxed{B_y = -\frac{\mu_0 K_0}{2}\frac{z}{|z|}}$$

b) Find $\vec{A}$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\rightarrow B_x = \frac{\partial}{\partial y}\cancel{A_z}^{0} - \frac{\partial}{\partial z}\cancel{A_y}^{0} = 0$$

$$B_y = \frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}\cancel{A_z}^{0} = -\frac{\mu_0 K_0}{2}\frac{z}{|z|}$$

$$B_z = \frac{\partial}{\partial x}\cancel{A_y}^{0} - \frac{\partial}{\partial y}A_x = 0$$

(i) $B_y \Rightarrow -\frac{\mu_0 K_0}{2} = \frac{\partial A_x}{\partial z}$, $z > 0$ ← possible a fcn of (x,y) as well

$$\Longrightarrow A_x = -\frac{\mu_0 K_0 z}{2} + \text{"}g(x,y)\text{"}$$

(ii) but $B_z = 0 = -\frac{\partial A_x}{\partial y} \rightarrow$ can't be a function of y

(iii) but $\vec{\nabla}\cdot\vec{A} = 0$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial \cancel{A_y}^{0}}{\partial y} + \frac{\partial \cancel{A_z}^{0}}{\partial z} = 0$$

↳ can't be a function of x

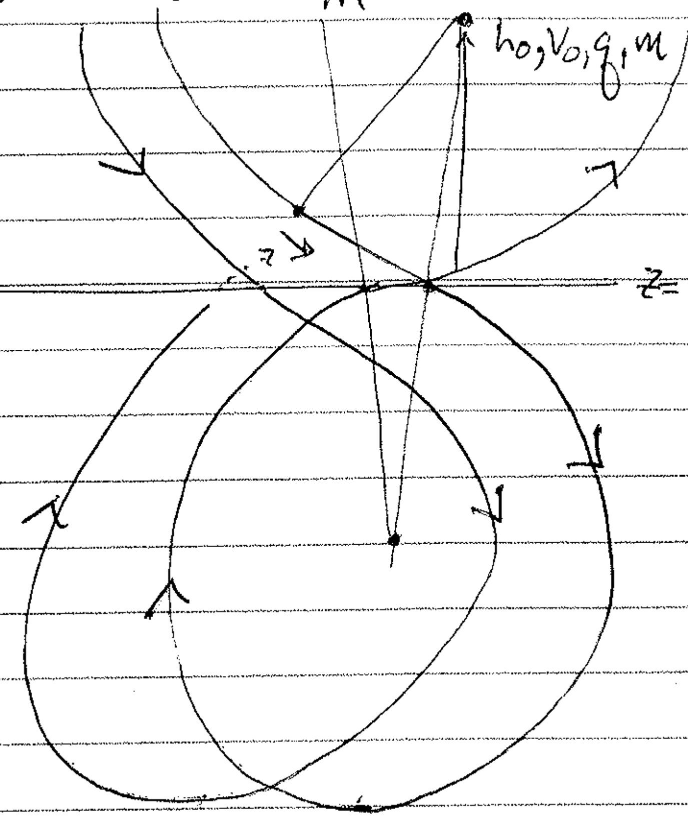
$$\Longrightarrow \boxed{A_x = -\frac{\mu_0 K_0}{2} z}$$

c) Coulomb Gauge, $\vec{\nabla}\cdot\vec{A} = 0$. It leads to $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ even if $\partial\vec{A}/\partial t \neq 0$

(so will not pass through sheet if $r_{gyro} < h_0$

$$\Rightarrow \text{if } \frac{mV_0}{qB} < h_0 \Rightarrow V_0 < \frac{qBh_0}{m}$$

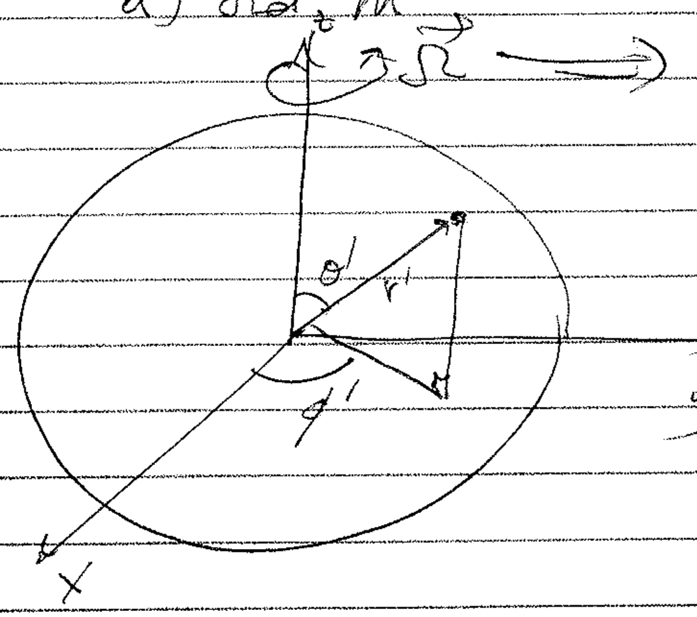
d)



makes cyclotron motion above and below $z=0$, w/center drifting to left.

Question 3

a) find $\vec{m}$



$\longrightarrow \vec{v} = \vec{\Omega} \times \vec{r}\,' = \Omega_0 (0,0,1) \times (x', y', z')$

where $(x', y', z') = (r'\sin\theta'\cos\phi', r'\sin\theta'\sin\phi', r'\cos\theta')$

$\Rightarrow \vec{v} = \Omega_0 (-y', x', 0)$

and $\vec{J} = \rho \vec{v} = \rho \Omega_0 (-y', x', 0)$

$$\vec{m} = \frac{1}{2}\int \vec{r}\,' \times \vec{J}(\vec{r}\,')\, d^3x'$$

$$= \frac{1}{2}\int (x', y', z') \times \rho \Omega_0 (-y', x', 0)\, d^3x'$$

$$= \frac{1}{2}\int \rho \Omega_0 (-x'z', -z'y', x'^2 + y'^2)\, dx'$$

<u>note</u>: x', y' contain $\sin\phi'$ & $\cos\phi'$ to 1st order $\Rightarrow$ they integrate to 0 over ϕ

$$= \frac{1}{2}\int C_\alpha r'^\alpha \Omega_0 (0, 0, r'^2 \sin^2\theta')\, r'^2 dr' \sin\theta' d\theta' d\phi'$$

$$= \frac{1}{2} C_\alpha \Omega_0 \int r'^{4+\alpha} \sin^3\theta'\, d\theta'\, d\phi'$$

$$= \cancel{\tfrac{1}{2}} C_\alpha \Omega_0 \left[\cancel{2}\pi \frac{R^{5+\alpha}}{5+\alpha} \int \sin^3\theta'\, d\theta' \right] \qquad \int (1 - \cos^2\theta')\, d\cos\theta' = 4/3$$

$$\boxed{\vec{m}_\alpha = \frac{4\pi}{3} C_\alpha \Omega_0 \frac{R^{5+\alpha}}{5+\alpha} \hat{z}}$$

$$\alpha = 0 \longrightarrow m_{z,0} = \frac{\cancel{4\pi}}{\cancel{3}}\left(\frac{\cancel{3}}{\cancel{4\pi}} \frac{Q}{R^3}\right) \Omega_0 \frac{R^5}{5} = \frac{1}{5} Q \Omega_0 R^2$$

$$\alpha = 2 \longrightarrow m_{z,2} = \frac{\cancel{4\pi}}{3}\left(\frac{5}{\cancel{4\pi}} \frac{Q}{R^5}\right) \Omega_0 \frac{R^7}{7} = \frac{5}{21} Q \Omega_0 R^2$$

b) find $\vec{B}$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{\vec{J}(\vec{r}')d^3x' \times \hat{\mathscr{r}}}{\mathscr{r}^2} \longleftarrow \vec{\mathscr{r}} = \cancel{\vec{r}}^{\,0} - \vec{r}' = -\vec{r}'$$

$$= \frac{\mu_0}{4\pi}\rho\Omega_0 \frac{(-y', x', 0)\times(-x', -y', -z')}{r'^3} d^3x'$$

integrate to 0 over ϕ

$$= \frac{\mu_0}{4\pi}\rho\Omega_0 \frac{(\cancel{-x'z'}, \cancel{y'z'}, -x'^2 - y'^2)}{r'^3} d^3x'$$

$$B_z = \frac{\mu_0}{4\pi}\Omega_0 \int \rho\left(\frac{-r'^2 \sin^2\theta'}{r'^3}\right) r'^2 dr' \sin\theta' d\theta' d\phi'$$

$$= \frac{\mu_0 \Omega_0 C_\alpha}{4\pi}\int r'^{(1+\alpha)} \sin^3\theta' d\theta' dr' d\phi'$$

$$= \frac{\mu_0 \Omega_0 C_\alpha}{\cancel{2}\,\cancel{4\pi}} \frac{R^{2+\alpha}}{2+\alpha}\left(\frac{\cancel{4}^{\,2}}{3}\right)(\cancel{2\pi})$$

$$\boxed{B_z = \frac{2\mu_0}{3}\Omega_0 C_\alpha \frac{R^{2+\alpha}}{2+\alpha}}$$

$$\alpha = 0 \rightarrow B_z = \frac{\cancel{2}\mu_0}{\cancel{3}}\Omega_0\left(\frac{\cancel{3}}{4\pi}\frac{Q}{R^3}\right)\frac{R^2}{\cancel{2}} = \frac{\mu_0}{4\pi}\frac{Q\Omega_0}{R}$$

$$\alpha = 2 \rightarrow B_z = \frac{2\mu_0}{3}\Omega_0\left(\frac{5}{4\pi}\frac{Q}{R^5}\right)\frac{R^4}{4} = \frac{\mu_0}{4\pi}\frac{Q\Omega_0}{R}\left(\frac{5}{6}\right)$$

c) $\vec{m}_\alpha \rightarrow$? for $\alpha = \infty \rightarrow m_\infty = \frac{\cancel{4\pi}}{3}\left(\frac{3+\alpha}{\cancel{4\pi}}\frac{Q}{R^{3+\alpha}}\right)\Omega_0 \frac{R^{5+\alpha}}{5+\alpha}$

$$\simeq \frac{1}{3}Q\Omega_0 R^2$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{Q\Omega_0 R^2}{3}\right)\left(\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right)}$$