Chapter 6.1 - 6.3

Triple Products of Vectors can be Defined

Given $\vec{A}, \vec{B}, \vec{C}$

X (1) $(\vec{A} \cdot \vec{B}) \cdot \vec{C}$, not defined

(2) $(\vec{A} \cdot \vec{B})\vec{C}$

$\vec{A}(\vec{B} \cdot \vec{C})$

$(\vec{A} \cdot \vec{B})\vec{C} \neq \vec{A}(\vec{B} \cdot \vec{C})$

(3) $\vec{A} \cdot (\vec{B} \times \vec{C}) = [\vec{A}\vec{B}\vec{C}]$

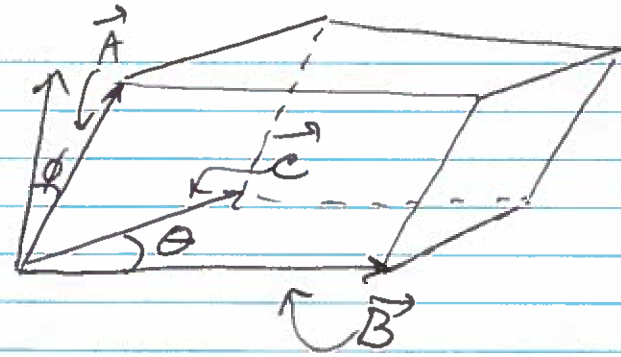
"Triple Scalar Product"

(4) $\vec{A} \times \vec{B} \times \vec{C}$

"Triple Vector Product"

Triple Scalar Product

(i) $\vec{A}\cdot(\vec{B}\times\vec{C})$



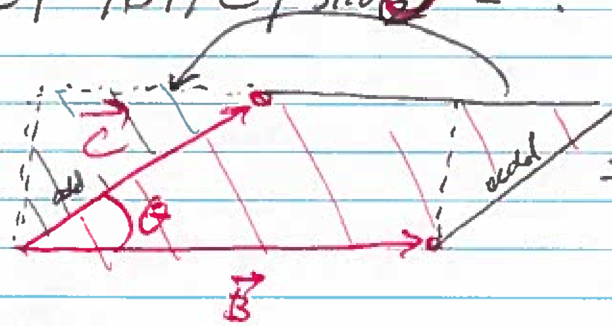
Geometric Interpretation of Triple Scalar Product

(a) $|\vec{B}\times\vec{C}| = |\vec{B}||\vec{C}|\sin\theta = ?$

triangle

$$\text{Area} = \left(\frac{C\cos\theta\, C\sin\theta}{2}\right)\times 2 + (C\sin\theta)(B - C\cos\theta) = BC\sin\theta$$

Base of figure



$\Rightarrow |\vec{B}\times\vec{C}|$ = area of parallelogram shown

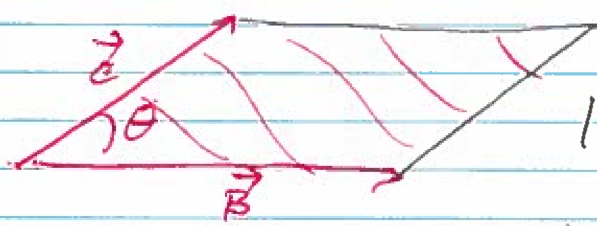
(b) the height of the parallelopiped is $|\vec{A}|\cos\phi$

$$\Rightarrow \vec{A}\cdot(\vec{B}\times\vec{C}) = |\vec{A}||\vec{B}\times\vec{C}|\cos\phi$$

$$= |\vec{A}||\vec{B}||\vec{C}|\sin\theta\cos\phi$$

$\equiv$ Volume of the parallelopiped!

Comment

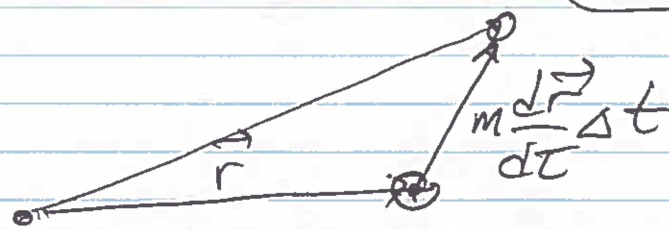


$|\vec{B}\times\vec{C}|$ = Area of parallelogram

$\Rightarrow$

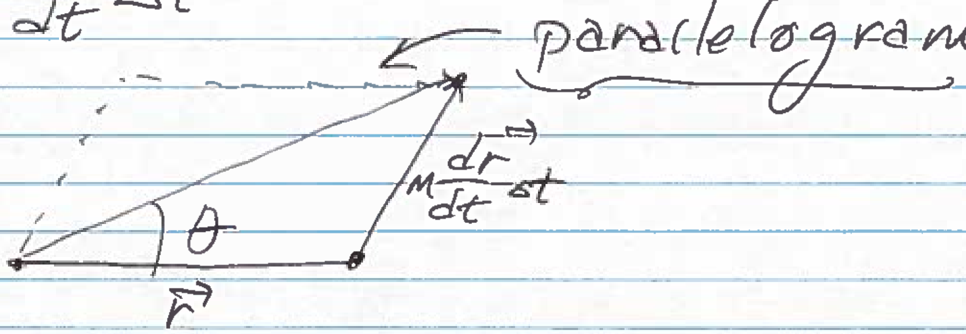
$\frac{1}{2}|\vec{B}\times\vec{C}|$ = Area of triangle

a) $\vec{\ell} = \vec{r} \times \vec{p} = \vec{r} \times m\dfrac{d\vec{r}}{dt}$ $\boxed{\text{angular momentum}}$



In the interval Δt, planet moves $\dfrac{d\vec{r}}{dt}\Delta t$

note:



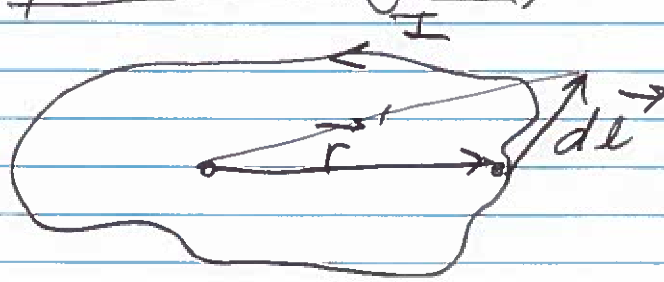
$$\vec{\ell} = m\underbrace{(\vec{r}\times\Delta\vec{r})}_{\text{area of parallelogram} \,=\, 2\times \text{triangle area}} \text{ in interval } \Delta t$$

Kepler's 2nd law: planet sweeps out equal areas in equal time intervals.

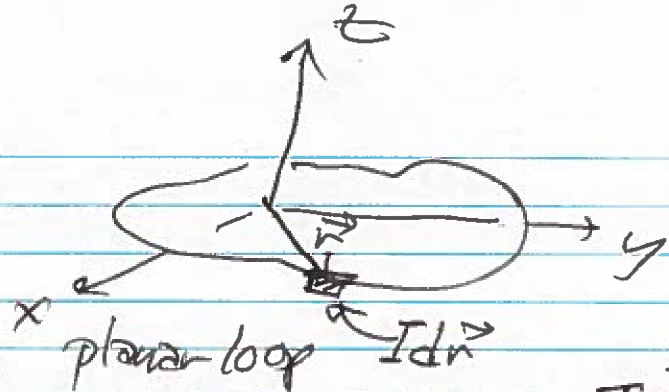
$\Rightarrow$ for fixed Δt, $\vec{r}\times\Delta r$ fixed $\Rightarrow$ $\vec{\ell}$ fixed!

b) planar surface, $\vec{m} = \dfrac{1}{2} I \oint \underbrace{\vec{r}\times d\vec{\ell}}_{\text{area element swept out}}$

$\longrightarrow$ around a loop $\Rightarrow$ Area of loop!

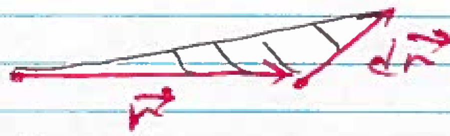


magnetic moment of



$$\vec{m} = \frac{1}{2}\int \vec{r}\times I d\vec{r} = \frac{1}{2} I \int \vec{r}\times d\vec{r} = I\vec{A}$$

because



$$\Longrightarrow \frac{1}{2}\int \vec{r}\times d\vec{r}$$

sweeps out the area of the triangle

(ii) $\vec{A}\cdot(\vec{B}\times\vec{C}) \equiv \boxed{[\vec{A}\vec{B}\vec{C}]}$

Let's show something interesting now (and then expand on the result)

ⓐ $\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

$$= \hat{i}(B_yC_z - B_zC_y) + \hat{j}(B_zC_x - B_xC_z) + \hat{k}(B_xC_y - B_yC_x)$$

ⓑ $\vec{A}\cdot(\vec{B}\times\vec{C}) = A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x)$

which we now notice says that we can write

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Suppose

$$(\vec{A}\times\vec{B})\cdot\vec{C} = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$= \vec{C}\cdot(\vec{A}\times\vec{B})$

Interchange "•" and "×" does not change the sign of the triple scalar product!

$$\rightarrow \vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

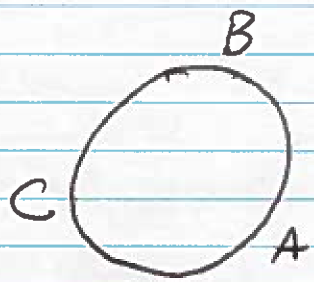
(iii) $\vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{A}\cdot(\vec{C}\times\vec{B})$

$\vec{C}\cdot(\vec{A}\times\vec{B}) = -\vec{C}\cdot(\vec{B}\times\vec{A})$

Triple Prod. ① & ② Griffiths

$\vdots$

<u>Construct</u>



cyclic permutation of terms in triple scalar product have the same magnitude & sign if ↻ or ↺ traversed

CW ↻

$\cancel{\vec{B}\cdot(\vec{A}\times\vec{C}) = \vec{A}\cdot(\vec{C}\times\vec{B}) = \vec{C}\cdot(\vec{B}\times\vec{A})}$

$$\vec{A}\cdot(\vec{C}\times\vec{B}) = \vec{C}\cdot(\vec{B}\times\vec{A}) = \vec{B}\cdot(\vec{A}\times\vec{C})$$

CCW ↺

$$\underbrace{\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(A\times\vec{B})}_{-\vec{A}\cdot(\vec{C}\times\vec{B})}$$

make a HW problem

Example of $[\vec{A}\vec{B}\vec{C}]$

Consider the vectors $\vec{A}$, $\vec{B}$, & $\vec{C}$; $\vec{A}$, $\vec{B}$, & $\vec{C}$ are not all parallel to the same plane,

$$[\vec{A}\vec{B}\vec{C}] = \vec{A}\cdot(\underbrace{\vec{B}\times\vec{C}}_{\perp \text{ to } \vec{B} \,\&\, \vec{C}}).$$

$\Rightarrow [\vec{A}\vec{B}\vec{C}] \neq 0$ unless $\vec{A}$ parallel to the same plane as $\vec{A}, \vec{B}$

> Show that any vector can be expressed as a linear combination of $\vec{A}$, $\vec{B}$, & $\vec{C}$ (for which $[\vec{A}\vec{B}\vec{C}] \neq 0$)

Sol^n

Write $\vec{V} = a\vec{A} + b\vec{B} + c\vec{C}$

(i) take $\vec{V}\times\vec{B} = a(\vec{A}\times\vec{B}) + b\cancel{(\vec{B}\times\vec{B})}^{0} + c(\vec{C}\times\vec{B})$

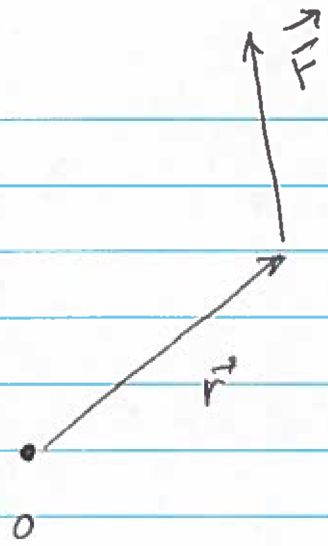
(ii) take $(\vec{V}\times\vec{B})\cdot\vec{C} = a(\vec{A}\times\vec{B})\cdot\vec{C} + c\cancel{(\vec{C}\times\vec{B})\cdot\vec{C}}^{0}$ ($\perp$ to $\vec{C}$ & $\vec{B}$)

$$\Rightarrow \boxed{a = \frac{[\vec{V}\vec{B}\vec{C}]}{[\vec{A}\vec{B}\vec{C}]}}$$

iii) Similarly

$$\boxed{b = \frac{[\vec{A}\vec{V}\vec{C}]}{[\vec{A}\vec{B}\vec{C}]}, \quad c = \frac{[\vec{A}\vec{B}\vec{V}]}{[\vec{A}\vec{B}\vec{C}]}}$$

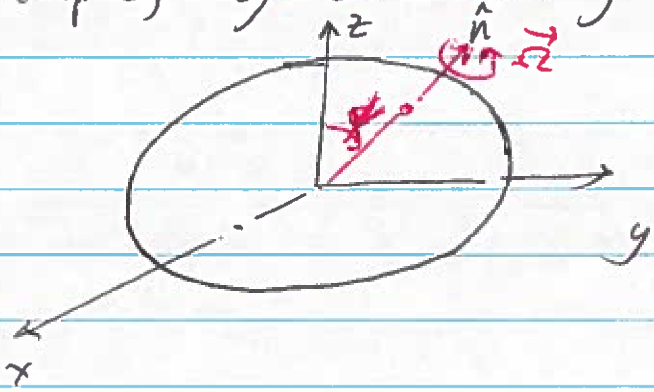
Torque



Torque about point O is

$$(\vec{r} \times \vec{F})$$

Suppose we want torque about an arbitrary line? For example, say we have an object (like the ⊕) where



the body axes are such but the spin axis is not parallel to the (a) body axis.

To get the torque about the spin axis, note that gravity is symmetric about z, find

$$\hat{n} \cdot (\vec{r} \times \vec{F})$$

Triple Vector Product

$$\vec{A} \times \underbrace{(\vec{B} \times \vec{C})}_{\perp \text{ to } \vec{B} \,\&\, \vec{C}} \stackrel{?}{=} (\vec{A} \times \vec{B}) \times \vec{C}$$

Q: Is Triple Vector Product associative?
(Cross product associative?)

A: Easily shown, but, for example, let

$$\vec{A} = \hat{i}, \quad \vec{B} = \hat{i}, \quad \vec{C} = \hat{j}$$

(i) $\vec{A} \times \vec{B} = \hat{i} \times \hat{i} = 0$

$$\rightarrow \boxed{(\vec{A} \times \vec{B}) \times \vec{C} = 0}$$

(ii) $\vec{B} \times \vec{C} = \hat{i} \times \hat{j} = \hat{k}$

$$\boxed{\rightarrow \vec{A} \times (\vec{B} \times \vec{C}) = \hat{i} \times \hat{k} = -\hat{j}}$$

$$\Rightarrow \vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

Cross product are not associative

BAC − CAB Rule

(i) $\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$, but what is $\vec{A}\times(\vec{B}\times\vec{C})$?

(ii) Write

$$(\vec{B}\times\vec{C}) = (b_2c_3 - b_3c_2,\ b_3c_1 - b_1c_3,\ b_1c_2 - b_2c_1)$$

(iii) → find $\vec{A}\times(\vec{B}\times\vec{C})$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \Big(a_2[b_1c_2 - b_2c_1] - a_3[b_3c_1 - b_1c_3],\ a_3[b_2c_3 - b_3c_2] - a_1[b_1c_2 - b_2c_1],\ a_1[b_3c_1 - b_1c_3] - a_2[b_2c_3 - b_3c_2]\Big)$$

Comment: in $\hat{i}$ component; $b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3)$

add $b_1a_1c_1$ subtract $b_1a_1c_1$

$$= b_1(\vec{A}\cdot\vec{C}) - c_1(\vec{A}\cdot\vec{B})$$

similarly for $\hat{j}$ & $\hat{k}$,

$$\cancel{\Rightarrow \vec{A}\times(\vec{B}\times\vec{C}) = \left([b_1 - c_1]\vec{A}\cdot\vec{B},\ [b_2 - c_2]\vec{A}\cdot\vec{B},\ [b_3 - c_3]\vec{A}\cdot\vec{B}\right)}$$

$$\Rightarrow \boxed{\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}$$

Eq (2), Griffiths

Look at:

$$(\vec{A}\times\vec{B})\times\vec{C} = -\vec{C}\times(\vec{A}\times\vec{B})$$

$$= -\vec{A}(\vec{C}\cdot\vec{B}) - (-\vec{B})(\vec{C}\cdot\vec{A})$$

$$= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{A}(\vec{C}\cdot\vec{B}) \quad \checkmark$$

$$\boxed{\begin{array}{c} \vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \\ \underline{\text{vs.}} \\ (\vec{A}\times\vec{B})\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{A}(\vec{C}\cdot\vec{B}) \end{array}}$$

grouping cross products differently

Given: Triple Scalar Products, Triple Vector Products, Dot products, Cross products can integrate multiplications of more vectors more easily.

① $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})$

triple scalar product

exchange "$\cdot$" and "$\times$"

$$\Rightarrow [(\vec{A}\times\vec{B})\times\vec{C}]\cdot\vec{D}$$

$$\vec{D}\cdot\left[\vec{B}(\vec{A}\cdot\vec{C}) - \vec{A}(\vec{C}\cdot\vec{B})\right]$$

$$= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{B})(\vec{A}\cdot\vec{D})$$

$$\Rightarrow \boxed{(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{B})(\vec{C}\cdot\vec{B})}$$

Lagrange's Identity

② $(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D})$

$$= -(\vec{C}\times\vec{D})\times(\vec{A}\times\vec{B})$$

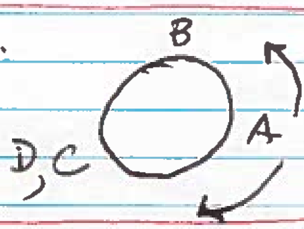
"BAC − CAB" product where $\vec{A}\times\vec{B}\to\vec{A}$, $\vec{C}\to\vec{B}$, $\vec{D}\to\vec{C}$

$$\overset{\vec A}{(\vec A \times \vec B)} \times (\overset{\vec B}{\vec C} \times \overset{\vec C}{\vec D}) = \vec C\left[(\vec A\times\vec B)\cdot\vec D\right] - \vec D\left[(\vec A\times\vec B)\cdot\vec C\right]$$

$$= \vec C\,\underbrace{[\vec D\vec A\vec B]} - \vec D\,\underbrace{[\vec C\vec A\vec B]}$$

triple scalar product

recall:



cyclic permutation for triple scalar product

$$\Rightarrow [\vec D\vec A\vec B] = [\vec A\vec B\vec D] = [\vec B\vec D\vec A]$$

and

$$[\vec C\vec A\vec B] = [\vec A\vec B\vec C] = [\vec B\vec C\vec A]$$

$$\boxed{(\vec A\times\vec B)\times(\vec C\times\vec D) = [\vec A\vec B\vec D]\vec C - [\vec A\vec B\vec C]D}$$

$$= [\vec C\vec D\vec A]\vec B - [\vec C\vec D\vec B]\vec A$$

Comment: [ ] are scalars $\Rightarrow$ $(\vec A\times\vec B)\times(\vec C\times\vec D)$ is a vector in plane defined by $\vec C$ & $\vec D$ (if $\vec C, \vec D$ are linearly independent)

and

also is a vector in plane defined by $\vec A$ & $\vec B$ (if $\vec A, \vec B$ are linearly independent)

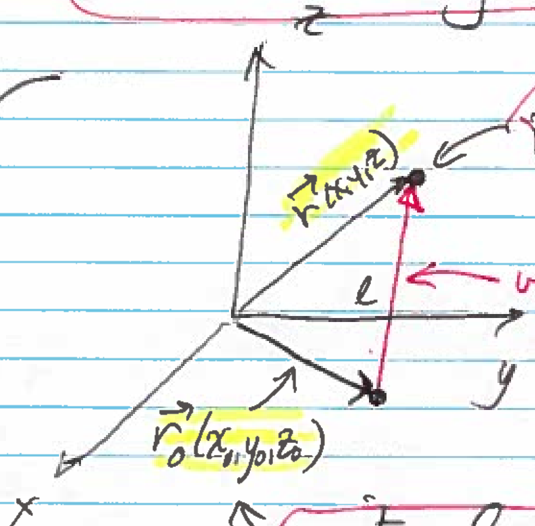
$\Rightarrow$ $(\vec A\times\vec B)\times(\vec C\times\vec D)$ is the intersection between planes AB, CD

Examples, Applications

Lines and Planes

line: Find the vector equation and a set of parametric scalar equations for the line through point $P_0(x_0, y_0, z_0)$ in the direction of vector

$$\boxed{\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \; ; \; \hat{i}, \hat{j}, \hat{k} \text{ are unit vectors}}$$



position of particle (along trajectory) as measured from the origin.

vector in direction of $\vec{A}$

position of particle at $t = 0$

the equation for the line is

$$\underbrace{(\vec{r} - \vec{r}_0)}_{0 \text{ at } t=0} = \vec{A}t$$

← t is a scalar parameter

for each point on trajectory l, there is a real $t \ni \vec{r} - \vec{r}_0 = \vec{A}t$

Component by component yields

(where $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$) → (1) $x - x_0 = a_1 t, \; y - y_0 = a_2 t, \; z - z_0 = a_3 t$

if $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0$, solve for t and then equate

$$\Longrightarrow \boxed{\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}}$$

Comment in 2-Dim

$$\frac{x-x_0}{a_1} = \frac{y-y_0}{a_2} \Rightarrow \frac{y-y_0}{x-x_0} = \frac{a_2}{a_1}$$

and

$$y = y_0 + \left(\frac{a_2}{a_1}\right)(x-x_0)$$

$$\boxed{y = \left[y_0 - \frac{a_2}{a_1}x_0\right] + \frac{a_2}{a_1}x}$$

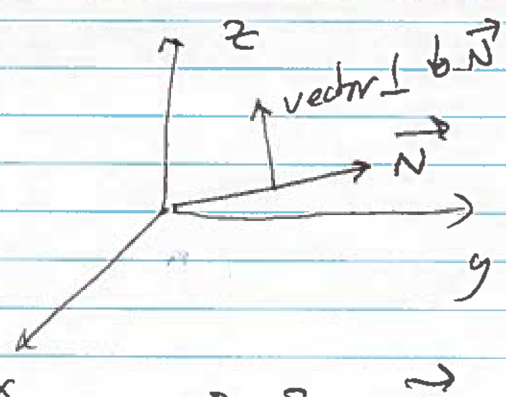
$\Rightarrow$ equation of a line where

$$y_0 - \frac{a_2}{a_1}x_0 \equiv \text{Intercept}$$

$$\frac{a_2}{a_1} \equiv \text{slope}$$

plane:

Suppose we have a vector, $\vec{N} = a\hat{i} + b\hat{j}$, find the equation of a line $\perp$ to $\vec{N}$.



define:
$\vec{r} - \vec{r}_0 \equiv \text{vector} = \vec{A}t$

$\underline{\underline{L}}: \Rightarrow (\vec{r} - \vec{r}_0) \cdot \vec{N} = 0$ (from dot product)

$$a(x-x_0) + b(y-y_0) + 0 = 0$$

$$\Rightarrow \boxed{\left(\frac{y-y_0}{x-x_0}\right) = -\frac{a}{b} \quad \text{line} \perp \text{to } \vec{N}}$$

what about z?

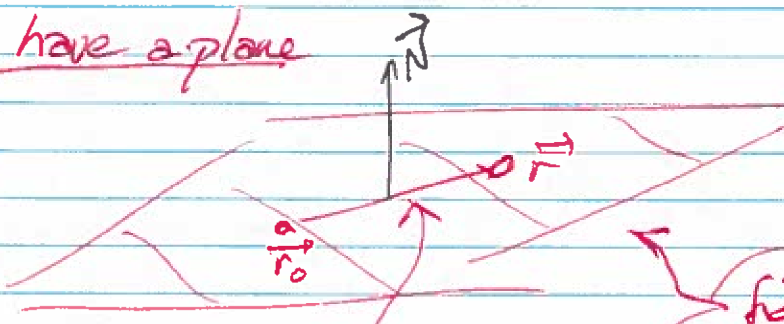
Example 1, p.109

find the equation of the plane that passes through the points

$$\underset{\vec{A}}{(-1,1,1)},\ \underset{\vec{B}}{(2,3,0)},\ \underset{\vec{C}}{(4,1,-2)}$$

Sol^n: we have a plane

define the plane



$\vec{N}$ is $\perp$ to $(\vec{r}-\vec{r}_0)$

$\vec{N} = a\hat{i} + b\hat{j} + c\hat{k}$

find eqn. for plane

$\vec{r}-\vec{r}_0$ lies in the plane

~~Find normal $\vec{N}$~~

we have $(\vec{r}-\vec{r}_0)\cdot\vec{N} = 0$; by def^n

$$\Rightarrow (x-x_0)a + (y-y_0)b + (z-z_0)c = 0$$

or gather constants,

$$xa + yb + zc = d = (ax_0 + by_0 + cz_0)$$

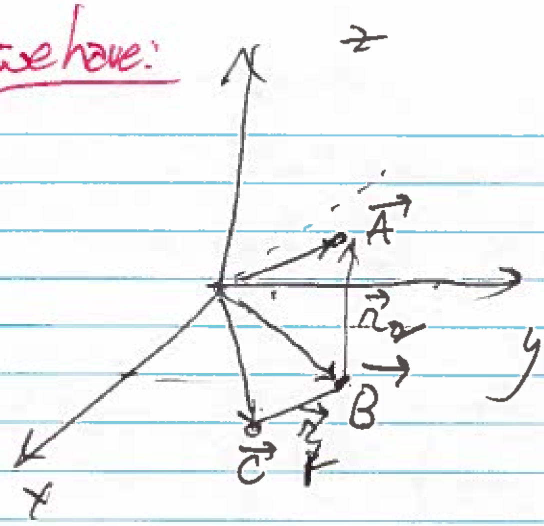
equation for a plane

equation of plane

in this class, for now, we take $d > 0$. d can be < 0 however.

find the particular plane

we have:



construct vectors w/ pts

$$\vec{r}_1 = \vec{C} - \vec{B}$$
&
$$\vec{r}_2 = \vec{A} - \vec{B}$$
lie in the desired plane

$$\boxed{\vec{r}_1 \times \vec{r}_2 \rightarrow \text{vector} \perp \text{to both } \vec{r}_1 \text{ \& } \vec{r}_2 \Rightarrow \text{Normal}, \vec{N}}$$

(i) $\vec{r}_1 = (\vec{C} - \vec{B}) = (-2, -2, -2)$

$\vec{r}_2 = (\vec{A} - \vec{B}) = (-3, -2, 1)$

and
$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -2 & -2 \\ -3 & -2 & 1 \end{vmatrix} = \hat{i}(-6) + \hat{j}(+8) + \hat{k}(-2)$$

$$\boxed{\vec{N} = (-6, +8, -2)}$$

Plug into plane equation,

$$(x - x_0)(-6) + (y - y_0)(+8) + (z - z_0)(-2) = 0$$

$$\boxed{3(x - x_0) - 4(y - y_0) + (z - z_0) = 0}$$

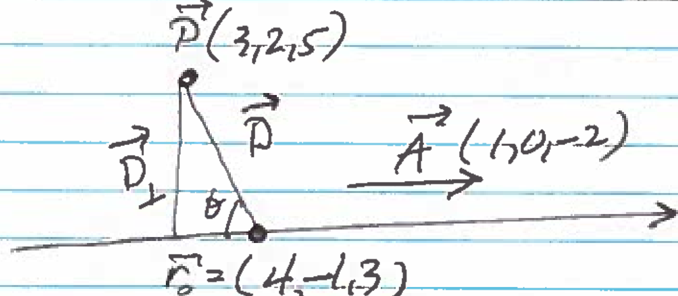
and then use $\vec{A}, \vec{B}, \vec{C}$ to find (x_0, y_0, z_0)

Example: 3.5-36

$\boxed{\text{Find the distance from } \vec{P} = (3,2,5) \text{ to the line passing through } (4,-1,3) \text{ parallel to } \vec{A} = (1,0,-2)}$

Sol^n

Geometry



$\vec{A} = (1,0,-2)$

(i) find $\vec{r}_0$ from $\boxed{(\vec{r} - \vec{r}_0) = \vec{A}t}$

at $t=0$, $\vec{r}_0 = (4,-1,3)$ from $\vec{r} = (4,-1,3)$

and $\boxed{\vec{r} - (4,-1,3) = (1,0,-2)t}$

(ii) find another point on the line, say at $t=1$.

at $t=1$, $\vec{r} = (5,-1,1)$

(iii) find $\vec{D} = \vec{P} - \vec{r} = (3,2,5) - (5,-1,1)$

$= (-2,3,4) \Rightarrow |\vec{D}| = \sqrt{29}$

(iv) find $\vec{D} \times \hat{A} = (-2,3,4) \times \frac{(1,0,-2)}{\sqrt{5}}$

$= \frac{1}{\sqrt{5}}(-6,0,-3) \rightarrow |\vec{D} \times \hat{A}| = \frac{3\cancel{\sqrt{5}}}{\cancel{\sqrt{5}}}$

(v) $|\vec{D} \times \hat{A}| = |\vec{D}|(1)\sin\theta = \boxed{D_\perp = 3}$