Homework 4

Due: 31 October 2016, by end of day

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25. page 323, 6.10, Problem 10
26. page 334, 6.11, Problem 3
27. page 335, 6.11, Problem 16
28. page 336, 6.11, Problem 20
29. page 338, 6.12, Problem 28

HW4

25. 6.10.10

Evaluate $\int \vec{V}\cdot d\vec{S}$ over $x^2+y^2+z^2=9$, $z\geq 0$ if $\vec{V}=(y, xz, 2z-1)$.

(i) Use divergence theorem. find $\boxed{\vec{\nabla}\cdot\vec{V}=2}$

$$\rightarrow \int(\vec{\nabla}\cdot\vec{V})d^3x = \int 2\left(\frac{r^2dr\,d\theta\sin\theta\,d\phi}{2}\right)$$

$$= \frac{4\pi r^3}{3}\Big|_0^3 \longrightarrow 3\cancel{6}\pi$$

$\int \vec{\nabla}\cdot\vec{V}d^3x = \int \vec{V}\cdot d\vec{\sigma}$, from divergence theorem.

$$= \int_{\text{hemisphere}} \vec{V}\cdot d\vec{\sigma} + \int_{\text{disk}} \vec{V}\cdot d\vec{\sigma}$$

z=0

$$= \int_{\text{hemisphere}} \vec{V}\cdot d\vec{\sigma} - \underbrace{\int (2z-1)\,r\,dr\,d\phi}_{+\pi r^2|_0^3 = 9\pi}$$

$$\rightarrow \int_{\text{hemisphere}} \vec{V}\cdot d\vec{S} = 27\pi$$

26. 6.163

$$\int \vec{\nabla} \times (x^2, z^2, -y^2) \cdot d\vec{\sigma}$$

over sufrace $z = \cancel{4} - x^2 - y^2$ for $z > 0$

Stokes' theorem $\int \vec{\nabla} \times (x^2, z^2, -y^2) \cdot d\vec{\sigma} = \oint \vec{V} \cdot d\vec{r}$

around circle $4 = x^2 + y^2$ $\longrightarrow$ $= \oint (x^2, 0, -y^2) \cdot d\vec{r}$

$$= \oint \left[x^2 dx - y^2 \cancelto{0}{\hat{z} \cdot d\vec{r}} \right]$$

Use polar coordiates, $x = r\cos\phi \rightarrow dx = -r\sin\phi\, d\phi$

$r = $ const

$$= \int r^2 \cos\phi (-r\sin\phi\, d\phi)$$

$$= r^3 \int \cos^2\phi \sin\phi\, d\phi$$

$$= \frac{-r^3 \cos^3\phi}{3} \Big|_0^{2\pi}$$

$$= 0$$

27. 6.11.16

Divergence theorem is for a closed surface but Stokes' theorem is for an open surface.

28. 6.11.20

if $\vec{V} = \left(ze^{zy} + x\sin xz,\ x\cos(xz),\ -z\sin xz\right)$

find $\vec{A} \ni \vec{V} = \vec{\nabla}\times\vec{A}$

$$\vec{V} = \begin{cases} \hat{i}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) = \hat{i}\left(ze^{zy} + x\sin xz\right) \\ \hat{j}\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) = \hat{j}\left(x\cos xz\right) \\ \hat{k}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) = \hat{k}\left(-z\sin xz\right) \end{cases}$$

To find $\vec{A}$, note that b/c of gauge freedom, $\vec{A}' = \vec{A} + \vec{\nabla}f$, where f is any scalar function

both $\vec{A}'$ & $\vec{A}$ are viable forms for $\vec{V} = \vec{\nabla}\times\vec{A}$

(i) find a form for $\vec{A}$ that leads to $\vec{V} = \vec{\nabla}\times\vec{A}$.

We can then make any $\vec{A}'$ by adding $\vec{\nabla}f$ to the $\vec{A}$.

choose $A_x = 0$

$$\Rightarrow -\frac{\partial}{\partial x} A_z = x\cos xz \rightarrow \boxed{A_z = -\sin(xz) + g(y,z)}$$

and

$$\frac{\partial}{\partial x} A_y = -z\sin xz \rightarrow \boxed{A_y = \cos(xz) + h(y,z)}$$

plug into $\hat{i}$ component of curl.

$$\frac{\partial}{\partial y}\left[-\sin xz + g(y,z)\right] - \frac{\partial}{\partial z}\left[\cos xz + h(y,z)\right]$$

$$= ze^{zy} + x\sin xz$$

$$0 + \frac{\partial}{\partial y} g(y,z) + \cancel{x\sin xz} - \frac{\partial}{\partial z} h(y,z)$$

$$= ze^{zy} + \cancel{x\sin xz}$$

$$\frac{\partial}{\partial y}\left(g(y,z)\right) - \frac{\partial}{\partial z}\left(h(y,z)\right) = ze^{zy}$$

if $h(y,z) = 0$

$$\rightarrow g(y,z) = e^{zy} + e(x,z)$$

set $e(x,z) = 0$

$$\rightarrow \boxed{\vec{A} = \left(0, \cos xz, -\sin xz + e^{zy}\right)}$$

can work

29. 6.12.28

$\oint \vec{F}\cdot d\vec{r}$ around circle $x^2+y^2+2\cancel{x}=0$
where $\vec{F} = (y, -x, 0)$

Sol^n

(i) $x^2+y^2+2x=0 \rightarrow (x+1)^2+y^2+(-1)=0$

$$\boxed{(x+1)^2+y^2=1}$$

(ii) go to frame where $x' = x+1$, $y' = y$

(iii) $\oint (y', -(x'-1), 0)\cdot d\vec{r}\,'$

$$= \oint \left[y'dx' - (x'-1)dy' \right]$$

In translated frame, $x' = \cos\phi$, $y' = \sin\phi$
$dx' = -\sin\phi\, d\phi$, $dy' = \cos\phi\, d\phi'$

$$= \oint \left(-\sin\phi \sin\phi\, d\phi - [\cos\phi - 1]\cos\phi\, d\phi \right)$$

$$= \oint \left[-\sin^2\phi - \cos^2\phi + \cos\phi \right] d\phi$$

$$\oint \vec{F}\cdot d\vec{r} = -2\pi$$