

Homework 5

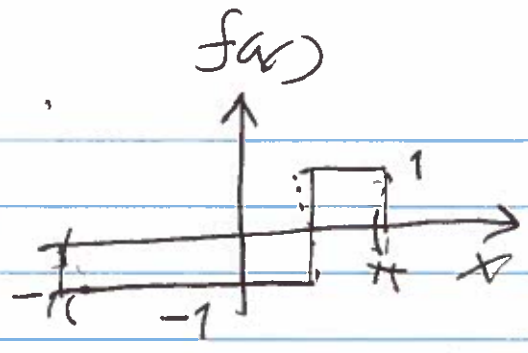
Due: 14 November 2016, by end of day

- 30. page 355, 7.5 Problem 4**
 - 31. page 360, 7.7 Problem 6**
 - 32. page 360, 7.7 Problem 12**
 - 33. page 363, 7.8 Problem 12**
 - 34. page 371, 7.10 Problem 24**
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HW5

7.5.4

$$f(x) = \begin{cases} -1 & -\pi < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a) \int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} \frac{a_0}{2} dx$$

$$-x \Big|_{-\pi}^{\pi/2} + x \Big|_{\pi/2}^{\pi} = \frac{a_0}{2} 2\pi = a_0 \pi$$

$$- \left(+\frac{\pi}{2} - (-\pi) \right) + \left(\pi - \frac{\pi}{2} \right) = \pi = a_0 \pi \rightarrow \boxed{a_0 = 1}$$

$$b) \int_{-\pi}^{\pi} f(x) \cos nx dx = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos nx dx$$

$$- \int_{-\pi}^{\pi/2} \cos nx dx + \int_{\pi/2}^{\pi} \cos nx dx = a_n \pi \delta_{nn'}$$

$$- \frac{\sin nx}{n'} \Big|_{-\pi}^{\pi/2} + \frac{\sin nx}{n'} \Big|_{\pi/2}^{\pi} =$$

~~$$- \frac{\sin n' \pi}{n'} + \frac{\sin n' \pi/2}{n'} + \frac{\sin n' \pi/2}{n'} - \frac{\sin n' (-\pi)}{n'}$$~~

n' odd

$$- \left[\frac{\sin n' \pi/2}{n'} \right] + \left[\frac{-\sin n' \pi/2}{n'} \right] =$$

$$- \frac{2}{n'} \sin n' \frac{\pi}{2} = -2, 0, \frac{2}{3}, 0, -\frac{2}{5}, \dots, n \text{ odd}$$

• 1 2 2 3 5 1

$$= -2 \sum_{\substack{n'=1 \\ \text{odd}}}^{\infty} \frac{(-1)^{\frac{n'-1}{2}}}{n'} = \pi a_n$$

$$\rightarrow a_n = -\frac{2}{n\pi} (-1)^{\frac{n-1}{2}}, n \text{ odd}$$

$$c) \int_{-\pi}^{\pi} f(x) \sin nx dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin nx dx$$

$$- \int_{-\pi}^{-\frac{\pi}{2}} \sin nx dx + \int_{\frac{\pi}{2}}^{\pi} \sin nx dx = \pi b_n d_n$$

$$+ \left[\frac{\cos nx}{n} \right]_{-\pi}^{-\frac{\pi}{2}} + \left[-\frac{\cos nx}{n} \right]_{\frac{\pi}{2}}^{\pi} =$$

$$\left(\frac{\cos \frac{n\pi}{2}}{n} - \frac{\cos n\pi}{n} \right) - \left(\frac{\cos n\pi}{n} - \frac{\cos \frac{n\pi}{2}}{n} \right) =$$

$$\frac{2}{n} \left(\cos \frac{n\pi}{2} - \cos n\pi \right) = + \frac{2}{1}, \frac{-1-1}{2}, \frac{2}{3}, \frac{1-1}{4}, \dots$$

$n'=1 \quad 2 \quad 3 \quad 4 \quad 5$

funny series
for b_n

$$\rightarrow f(x) = -\frac{1}{2} + \frac{2}{\pi} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\cos nx}{n} + \frac{2}{\pi} \left[\sin x - \frac{2 \sin 2x}{2} + \frac{2}{3} \sin 3x + \dots \right]$$

we could have ^{used} Prob 3 to give this solution.

Prob 3,

$$f_3(x) = \begin{cases} 0 & -\pi < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

if we subtract 1 from $f_3(x) \rightarrow$

$$\begin{cases} -1 & -\pi < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

so, if we then add $f_3(x)$ to the tabbed $f_3(x)$

$$f_3(x) + [f_3(x) - 1] = \begin{cases} -1 & -\pi < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

the $f(x)$ of 7.5.4 is found.

Linearity of Fourier series \rightarrow $f(x)$ of 7.5.4 can be constructed in this manner.

31 7.7.b

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ -1 & 0 < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Soln

$$\int_{-\pi}^{\pi} f(x) e^{-in'x} dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{inx} e^{-in'x} dx$$

$$-\int_0^{\pi/2} e^{-in'x} dx + \int_{\pi/2}^{\pi} e^{-in'x} dx = 2\pi c_{n'} \delta_{nn'}$$

$$\frac{e^{-in'x}}{in'} \Big|_0^{\pi/2} - \frac{e^{-in'x}}{in'} \Big|_{\pi/2}^{\pi} =$$

$$\left(\frac{e^{-in'\pi/2} - 1}{in'} \right) - \left(\frac{e^{-in'\pi} - e^{-in'\pi/2}}{in'} \right) =$$

$$\frac{2 e^{-in'\pi/2}}{in'} - \frac{1 + e^{-in'\pi}}{in'} =$$

$$-\frac{2i}{in'} (-1)^{\frac{n'+1}{2}} - \frac{2}{in'} (-1)^{\frac{n'+2}{2}}$$

n' odd

$$\frac{2}{in'} \text{ for } n' \text{ even}$$

→ form for series and

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\frac{2}{in} (1 - (-1)^{\frac{n-2}{2}}) e^{inx} \right]$$

even

$$+ \frac{1}{2\pi} \sum_{\substack{n=-\infty \\ \text{odd}}}^{\infty} \left[\frac{2}{n} (-1)^{\frac{n-1}{2}} e^{inx} \right]$$

Write out $f(x)$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^n e^{-inx} - \sum_{n=3}^{\infty} \frac{2}{n} (-1)^{\frac{n}{2}} e^{-3inx} - \sum_{n=5}^{\infty} \frac{2}{n} (-1)^{\frac{n}{2}} e^{-5inx} + \dots$$

$$+ \sum_{n=1}^{\infty} \frac{2}{n} e^{inx} + \sum_{n=3}^{\infty} \frac{2}{n} (-1)^{\frac{n}{2}} e^{3inx} + \sum_{n=5}^{\infty} \frac{2}{n} (-1)^{\frac{n}{2}} e^{5inx} + \dots$$

$$- \sum_{n=2}^{\infty} \frac{2}{i2} (1-1) e^{-2inx} - \sum_{n=4}^{\infty} \frac{2}{i4} (1+1) e^{-4inx} - \sum_{n=6}^{\infty} \frac{2}{i6} (1-1) e^{-6inx} + \dots$$

$$+ \sum_{n=2}^{\infty} \frac{2}{i2} (1+1) e^{2inx} + \sum_{n=4}^{\infty} \frac{2}{i4} (1+1) e^{4inx} + \sum_{n=6}^{\infty} \frac{2}{i6} (1-1) e^{6inx} + \dots$$

$$= 2 \left[e^{ix} \left(\frac{1}{1} \right) + \frac{1}{1} e^{-ix} + e^{3ix} \left(-\frac{1}{3} \right) + e^{-3ix} \left(-\frac{1}{3} \right) + e^{5ix} \left(\frac{1}{5} \right) + e^{-5ix} \left(\frac{1}{5} \right) + \dots \right]$$

$$+ 2 \left[\frac{2}{4i} (e^{4ix} - e^{-4ix}) + \dots \right]$$

$$= 2 \left[2 \cos x - \frac{2}{3} \cos 3x + \frac{2}{5} \cos 5x + \dots \right]$$

$$+ 2 \left[\sin 4x + \frac{1}{2} \sin 6x + \dots \right]$$

32. 7.1.12

for real $f(x)$, show that $c_{-n} = \overline{c_n}$ where $\overline{c_n}$ means complex conjugate of c_n

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\begin{aligned} a) f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{e^{inx} + e^{-inx}}{2} + \sum_{n=1}^{\infty} b_n \frac{e^{inx} - e^{-inx}}{2i} \end{aligned}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n + b_n i}{2} e^{inx}$$

$$+ \sum_{n=1}^{\infty} \frac{a_n - b_n i}{2} e^{-inx}$$

← rewrite

as $n = -\infty$

$$\rightarrow \sum_{n=-\infty}^{-1} \frac{a_n - b_n i}{2} e^{inx}$$

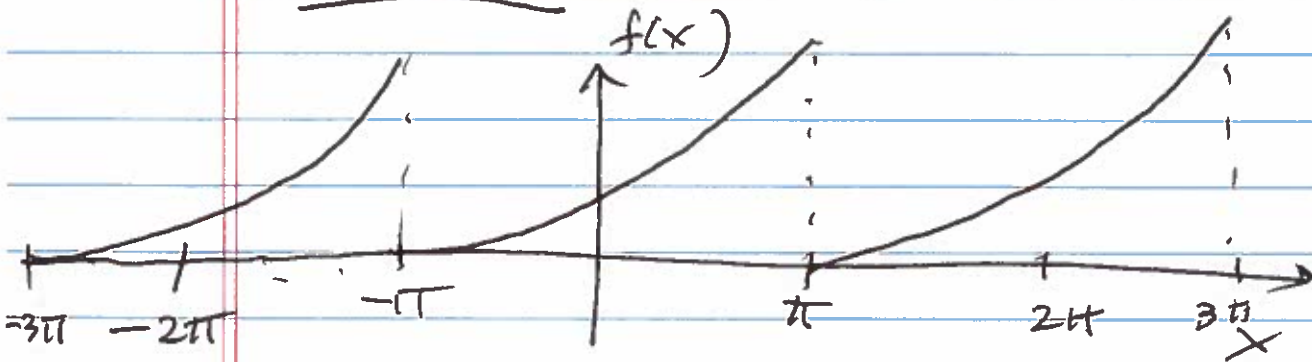
if $f(x)$ was real, a_n, b_n real

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n + \frac{1}{i} b_n}{2} e^{inx} + \sum_{n=-\infty}^{-1} \frac{a_n - \frac{1}{i} b_n}{2} e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}, \text{ where } c_{-n} = \overline{c_n}$$

33. 7.8.12

$$f(x) = e^x, \quad -\pi < x < \pi$$



$$a) f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx) + \sum_{n=1}^{\infty} (b_n \sin nx)$$

$$(i) \int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} e^x dx = e^x \Big|_{-\pi}^{\pi} = e^{\pi} - e^{-\pi}$$

$$= 2 \sinh \pi$$

$$(ii) \int_{-\pi}^{\pi} f(x) \cos n'x dx = \int_{-\pi}^{\pi} e^x \frac{e^{in'x} + e^{-in'x}}{2} dx$$

$$= \frac{e^{x+in'x}}{2(1+in')} \Big|_{-\pi}^{\pi} + \frac{e^{x-in'x}}{2(1-in')} \Big|_{-\pi}^{\pi}$$

$$= \frac{e^{\pi} (-1)^{n'}}{2(1+in')} - \frac{e^{-\pi} (-1)^{n'}}{2(1+in')}$$

$$+ \frac{e^{\pi} (-1)^{n'}}{2(1-in')} - \frac{e^{-\pi} (-1)^{n'}}{2(1-in')}$$

$$= \frac{(-1)^{n'}}{2} \left[\frac{e^{\pi} - e^{-\pi}}{1+in'} + \frac{e^{\pi} - e^{-\pi}}{1-in'} \right]$$

$$= (-1)^{n'} \sinh \pi \left[\frac{(1-in') + (1+in')}{1+n'^2} \right] = (-1)^{n'} \frac{2 \sinh \pi}{1+n'^2}$$

$$(iii) \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \int_{-\pi}^{\pi} e^{ix} \left(\frac{e^{inx} - e^{-inx}}{2i} \right) dx$$

$$= \frac{e^{x+inx}}{2i(1+i^n)} \Big|_{-\pi}^{\pi} - \frac{e^{x-inx}}{2i(1-i^n)} \Big|_{-\pi}^{\pi}$$

$$= \frac{e^{\pi} (-1)^{n'}}{2i(1+i^n)} - \frac{e^{-\pi} (-1)^n}{2i(1+i^n)} - \frac{e^{\pi} (-1)^n}{2i(1-i^n)} + \frac{e^{-\pi} (-1)^{n'}}{2i(1-i^n)}$$

$$= \frac{(-1)^{n'}}{2i} (e^{\pi} - e^{-\pi}) \left[\frac{1}{1+i^n} - \frac{1}{1-i^n} \right]$$

$$= \frac{(-1)^{n'}}{i} \sinh \pi \left[\frac{1-i^n - 1-i^n}{1+n^2} \right]$$

$$= - \frac{(-1)^{n'}}{1+n^2} 2n \sinh \pi$$

$$\Rightarrow a_0 = \frac{2}{\pi} \sinh \pi$$

$$a_n = \frac{2}{\pi} \sinh \pi \frac{(-1)^n}{1+n^2}$$

$$b_n = - \frac{2}{\pi} \sinh \pi \frac{(-1)^n}{1+n^2}$$

$$\Rightarrow f(x) = \frac{\sinh \pi}{\pi} \left[\sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2} \cos nx + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2} \sin nx \right]$$

+ 1]

$$b) f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\int_{-\pi}^{\pi} f(x) e^{-in'x} dx = 2\pi c_n \delta_{nn'}$$

$$\int_{-\pi}^{\pi} e^{x-in'x} dx =$$

$$\rightarrow \frac{e^{x-in'x}}{1-in'} \Big|_{-\pi}^{\pi} = \frac{e^{\pi} (-1)^{n'}}{1-in'} - \frac{e^{-\pi} (-1)^{n'}}{1-in'}$$

$$= \frac{(-1)^{n'}}{1-in'} (e^{\pi} - e^{-\pi})$$

$$= \frac{2}{1-in'} (-1)^{n'} \sinh \pi$$

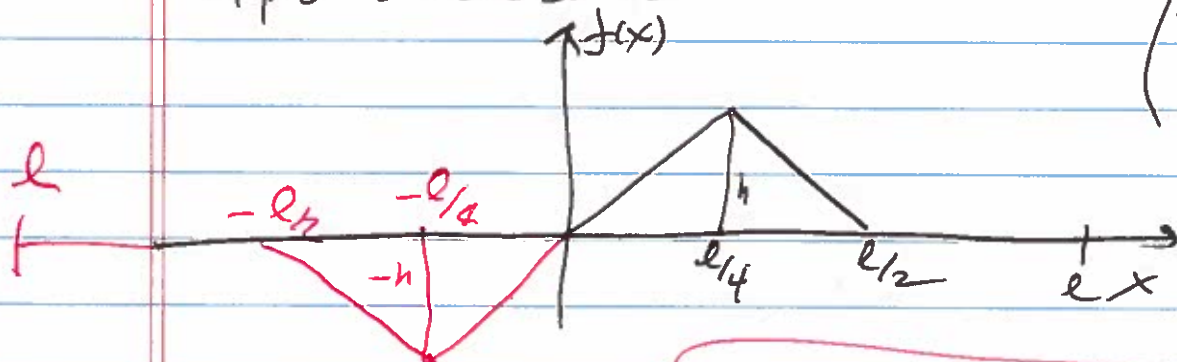
$$\Rightarrow f(x) = \sum_{n=-\infty}^{\infty} \frac{2}{1-in'} (-1)^{n'} \frac{\sinh \pi}{\pi} e^{inx}$$

$$= \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n'}}{1-in'} e^{inx}$$

34 7.10.24

Expand as a sine series

$$f(x) = \begin{cases} x \frac{4h}{l}, & 0 < x < \frac{l}{4} \\ 2h - x \frac{4h}{l}, & \frac{l}{4} < x < \frac{l}{2} \\ 0, & \frac{l}{2} < x < l \end{cases}$$



$$\Rightarrow \text{Period} = 2l \rightarrow \text{frequency} = \frac{2\pi n}{2l} = \frac{\pi n}{l}$$

$$\Rightarrow b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \left[\int_0^{\frac{l}{4}} x \frac{4h}{l} \sin\left(\frac{n\pi x}{l}\right) dx + \int_{\frac{l}{4}}^{\frac{l}{2}} \left(2h - x \frac{4h}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx \right]$$

$$= \frac{2}{l} \left[\frac{4h}{l} \frac{l}{n\pi} \frac{l}{n\pi} \int_0^{\frac{n\pi}{4}} \xi \sin \xi d\xi + \int_{\frac{l}{4}}^{\frac{l}{2}} 2h \sin\left(\frac{n\pi x}{l}\right) dx \right]$$

$$= \frac{2}{l} \left[\frac{4h l^2}{l n^2 \pi^2} \left(\sin \xi - \xi \cos \xi \right) \Big|_0^{\frac{n\pi}{4}} - \frac{4h l^2}{l n^2 \pi^2} \left(\sin \xi - \xi \cos \xi \right) \Big|_{\frac{n\pi}{4}}^{\frac{n\pi}{2}} \right]$$

$$+ \frac{2h}{\frac{n\pi}{l}} \left(-\cos\left(\frac{n\pi x}{l}\right) \right) \Big|_{\frac{l}{4}}^{\frac{l}{2}}$$

$$b_n = \frac{2}{l} \left[\frac{4hl}{n^2\pi^2} \left(\sin \frac{n\pi}{4} - \frac{n\pi}{4} \cos \frac{n\pi}{4} \right) - \left(\sin \frac{n\pi}{2} - \sin \frac{n\pi}{4} - \frac{n\pi}{2} \cos \frac{n\pi}{2} + \frac{n\pi}{4} \cos \frac{n\pi}{4} \right) \right]$$

$$= \frac{2hl}{n\pi} \left(\cos \frac{n\pi}{2} - \cos \frac{n\pi}{4} \right)$$

$$b_n = \frac{2}{l} \left[\frac{4hl}{n^2\pi^2} \left(2\sin \frac{n\pi}{4} - \sin \frac{n\pi}{2} - \frac{n\pi}{2} \cos \frac{n\pi}{4} + \frac{n\pi}{2} \cos \frac{n\pi}{2} \right) \right]$$

$$- \frac{2hl}{n\pi} \left(\cos \frac{n\pi}{2} - \cos \frac{n\pi}{4} \right)$$

$$b_n = \left[\frac{8h}{n^2\pi^2} \left(2\sin \frac{n\pi}{4} - \sin \frac{n\pi}{2} - \frac{n\pi}{2} \cos \frac{n\pi}{4} + \frac{n\pi}{2} \cos \frac{n\pi}{2} \right) \right]$$

$$- \frac{4h}{n\pi} \left(\cos \frac{n\pi}{2} - \cos \frac{n\pi}{4} \right)$$

$$= \frac{8h}{(n\pi)^2} \left[2\sin \frac{n\pi}{4} - \sin \frac{n\pi}{2} \right] + \frac{8h}{(n\pi)} \left[-\frac{1}{2} \cos \frac{n\pi}{4} + \frac{1}{2} \cos \frac{n\pi}{2} - \frac{1}{2} \cos \frac{n\pi}{2} + \frac{1}{2} \cos \frac{n\pi}{4} \right]$$

$$= \frac{8h}{(n\pi)^2} \left[2\sin \left(\frac{n\pi}{4} \right) - \sin \left(\frac{n\pi}{2} \right) \right]$$

first few terms

$$b_1 = \frac{8h}{\pi^2} \left(2\sin\frac{\pi}{4} - 1 \right) = \frac{16h}{\pi^2} \left(\sin\frac{\pi}{4} - \frac{1}{2} \right)$$

$$b_2 = \frac{8h}{4\pi^2} (2) = \frac{4h}{\pi^2}$$

$$b_3 = \frac{8h}{9\pi^2} \left(3\sin\frac{3\pi}{4} + 1 \right) = \frac{+8h}{3\pi^2} \sin\frac{\pi}{4} + 1$$

$$b_4 = \frac{8h}{16\pi^2} (0) = 0$$

$$b_5 = \frac{8h}{25\pi^2} \left(2\sin\frac{5\pi}{4} - 1 \right) = \frac{-16}{25\pi^2} \sin\frac{\pi}{4} - \frac{8h}{25\pi^2}$$

$$b_6 = \frac{8h}{36\pi^2} (-2) = \frac{-4h}{9\pi^2}$$

$$b_7 = \frac{8h}{49\pi^2} \left(2\sin\frac{7\pi}{4} + 1 \right) = \frac{-16h}{49\pi^2} \sin\frac{\pi}{4} + \frac{8h}{49\pi^2}$$

$$b_8 = \frac{8h}{64\pi^2} (0) = 0$$

and, in general,

$$f(x) = \sum_{n=1}^{\infty} \frac{16h}{\pi^2} \left(\frac{\sin\left(\frac{n\pi}{4}\right)}{n^2} - \frac{\sin\left(\frac{n\pi}{2}\right)}{2n^2} \right) \sin\left(\frac{n\pi x}{L}\right)$$

35 7.11.7

Sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using Prob 5.8

Given $f(x) = 1+x$, find

$$\rightarrow f(x) = 1 + 2 \left(\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots \right)$$

$$\begin{aligned} \text{a) } \int_{-\pi}^{2\pi} f(x) f(x) dx &= \int_{-\pi}^{\pi} (1+x)^2 dx = \int_{-\pi}^{\pi} (1+2x+x^2) dx \\ &= \left(x + x^2 + \frac{x^3}{3} \right) \Big|_{-\pi}^{\pi} \\ &= (2\pi) + (0) + \frac{2\pi^3}{3} \end{aligned}$$

$$\rightarrow \overline{|f(x)|}^2 = 1 + \frac{\pi^2}{3}$$

$$\text{b) } b_n = \frac{2}{n} (-1)^{n-1}, a_0 = 2$$

c) Parseval's theorem

$$\begin{aligned} \overline{|f(x)|}^2 &= \left(\frac{a_0}{2} \right)^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \\ &= 1 + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \right) \end{aligned}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \left[\overline{|f(x)|}^2 - 1 \right] \frac{1}{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6} \left(1 + \frac{\pi^2}{3} - 1 \right)$$

$$= \frac{\pi^2}{6}$$