Homework 5

Due: 14 November 2016, by end of day

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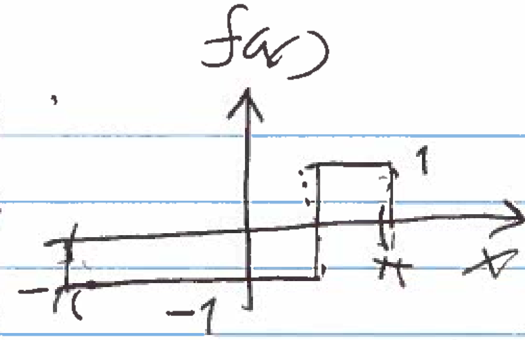
HW5

3) 7.5.4



$$f(x) = \begin{cases} -1 & -\pi < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

a) $\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx$

$$-x\Big|_{-\pi}^{\frac{\pi}{2}} + x\Big|_{\frac{\pi}{2}}^{\pi} = \frac{a_0}{2} 2\pi = a_0 \pi$$

$$-\left(+\frac{\pi}{2} - -\pi\right) + \left(\pi - \frac{\pi}{2}\right) = -\pi = a_0 \pi \rightarrow \boxed{a_0 = -1}$$

b) $\displaystyle\int_{-\pi}^{\pi} f(x) \cos n'x\,dx = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos n'x\,dx$

$$-\int_{-\pi}^{\frac{\pi}{2}} \cos n'x\,dx + \int_{\frac{\pi}{2}}^{\pi} \cos n'x\,dx = a_n \pi \delta_{nn'}$$

$$-\frac{\sin n'x}{n'}\Big|_{-\pi}^{\frac{\pi}{2}} + \frac{\sin n'x}{n'}\Big|_{\frac{\pi}{2}}^{\pi} =$$

n' odd

$$-\left[\frac{\sin n'\frac{\pi}{2}}{n'}\right] + \left[\frac{-\sin n'\frac{\pi}{2}}{n'}\right] =$$

$$-\frac{2}{n'} \sin n'\frac{\pi}{2} = -2, 0, \frac{2}{3}, 0, -\frac{2}{5}, \dots, n \text{ odd}$$

$$= -2\sum_{\substack{n'=1\\ n\,odd}}^{\infty} \frac{(-1)^{\frac{n'-1}{2}}}{n'} = \pi a_n \delta_{nn'}$$

$$\rightarrow \boxed{a_n = -\frac{2}{n\pi}(-1)^{\frac{n-1}{2}},\ n\ odd}$$

c) $$\int_{-\pi}^{\pi} f(x)\sin n'x\,dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin n'x\,dx$$

$$-\int_{-\pi}^{\frac{\pi}{2}} \sin n'x\,dx + \int_{\frac{\pi}{2}}^{\pi} \sin n'x\,dx = \pi b_n \delta_{nn'}$$

$$+\left[\frac{\cos n'x}{n'}\Big|_{-\pi}^{\pi/2}\right] + \left[-\frac{\cos n'x}{n'}\Big|_{\frac{\pi}{2}}^{\pi}\right] =$$

$$\left(\frac{\cos\frac{n'\pi}{2}}{n'} - \frac{\cos n'\pi}{n'}\right) - \left(\frac{\cos n'\pi}{n'} - \frac{\cos\frac{n'\pi}{2}}{n'}\right) =$$

$$\frac{2}{n'}\left(\cos\frac{n'\pi}{2} - \cos n'\pi\right) = +\frac{2}{1},\ \frac{-1-1}{2}2,\ \frac{2}{3},\ \cancel{1}\cancel{-1}\,0,\ \ldots$$

$n'=1 \quad 2 \quad 3 \quad 4 \quad 5$

funny series

for b_n

$$\boxed{\rightarrow f(x) = -\frac{1}{2}\cdot\frac{2}{\pi}\sum_{\substack{n=1,\\ odd}}^{\infty}(-1)^{\frac{n-1}{2}}\frac{\cos nx}{n} + \frac{2}{\pi}\left[\sin x - \frac{2\sin 2x}{2} + \frac{2}{3}\sin 3x + \cdots\right]}$$

we could have used Prob 3 to given this solution.

Prob 3,

$$f_3(x) = \begin{cases} 0 & -\pi < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

if we subtract 1 from $f_3(x) \rightarrow \begin{cases} -1 & -\pi < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$

so, if we then add $f_3(x)$ to the tasked $f_3(x)$

$$f_3(x) + [f_3(x) - 1] = \begin{cases} -1 & -\pi < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

the $f(x)$ of 7.5.4 is found.

Linearity of fourier series $\rightarrow$ $f(x)$ of 7.5.4 can be constructed in this manner.

3) 7.7.6

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ -1 & 0 < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Sol'n

$$\int_{-\pi}^{\pi} f(x) e^{-in'x} dx = \sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{-\pi}^{\pi} e^{inx} e^{-in'x} dx}$$

$$-\int_0^{\pi/2} e^{-in'x} dx + \int_{\frac{\pi}{2}}^{\pi} e^{-in'x} dx \qquad = 2\pi c_n \delta_{nn'}$$

$$\left.\frac{e^{-in'x}}{in'}\right|_0^{\frac{\pi}{2}} - \left.\frac{e^{-in'x}}{in'}\right|_{\frac{\pi}{2}}^{\pi} =$$

$$\left(\frac{e^{-in'\frac{\pi}{2}} - e^{i0}}{in'}\right) - \left(\frac{e^{-in'\pi} - e^{-in'\frac{\pi}{2}}}{in'}\right) =$$

$$\underbrace{2\frac{e^{-in'\pi/2}}{in'}} - \underbrace{\frac{1+e^{-in'\pi}}{in'}} =$$

$$-\frac{2i}{in'}(-1)^{\frac{n'-1}{2}} \qquad -\frac{2}{in'}(-1)^{\frac{n'}{2}} \qquad \frac{2}{in'} \text{ for } n' \text{ even}$$

n' odd $\qquad$ n' even

→ funny form for series cool

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty,\ \text{even}}^{\infty} \left[\frac{2}{in} \left(1 - (-1)^{\frac{n-2}{2}} \right) e^{inx} \right]$$

$$+ \frac{1}{2\pi} \sum_{n=-\infty,\ \text{odd}}^{\infty} \left[\frac{2}{n} (-1)^{\frac{n-1}{2}} e^{inx} \right]$$

Write out $f(x)$

$$f(x) = -\frac{2}{1}(-1)e^{-ix} - \frac{2}{3}(-1)^{-\frac{4}{2}}e^{-3ix} - \frac{2}{5}(-1)^{-\frac{6}{2}}e^{-5xi} - \cdots$$

$n=1$ $\quad n=-3$ $\quad n=-5$

$$+\frac{2}{1}e^{ix} + \frac{2}{3}(-1)^{\frac{2}{2}}e^{3ix} + \frac{2}{5}(-1)^{\frac{4}{2}}e^{i5x} + \cdots$$

$n=1$ $\quad n=3$ $\quad n=5$

$$-\frac{2}{i2}(\cancel{1}-1)^{0}e^{-2ix} - \frac{2}{i4}(1+1)e^{-4ix} - \frac{2}{i6}(\cancel{1}-1)^{0}e^{-6ix} - \cdots$$

$n=-2$ $\quad n=-4$ $\quad n=-6$

$$+\frac{2}{i2}\cancel{(1-1)}^{0}e^{2ix} + \frac{2}{i4}(1+1)e^{4ix} + \frac{2}{i6}\cancel{(1-1)}^{0}e^{6ix} + \cdots$$

$n=2$ $\quad n=4$ $\quad n=6$

$$= 2\left[e^{ix}\left(\frac{1}{1}\right) + \frac{1}{1}e^{-ix} + e^{3ix}\left(-\frac{1}{3}\right) + e^{-3ix}\left(-\frac{1}{3}\right) + e^{i5x}\left(\frac{1}{5}\right) + e^{-5ix}\left(\frac{1}{5}\right) + \cdots\right]$$

$$+2\left[\frac{2}{4i}\left(e^{4ix} - e^{-4ix}\right) + \cdots\right]$$

$$= 2\left[2\cos x - \frac{2}{3}\cos 3x + \frac{2}{5}\cos 5x + \cdots\right]$$

$$+2\left[\sin 4x + \frac{1}{2}\sin 6x + \cdots\right]$$

32. 7.1.12

for real $f(x)$, show that $c_{-n} = \overline{c_n}$ where $\overline{c_n}$ means complex conjugate of c_n

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

a) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{e^{inx} + e^{-inx}}{2} + \sum_{n=1}^{\infty} b_n \frac{e^{inx} - e^{-inx}}{2i}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n + b_n \frac{1}{i}}{2} e^{inx}$$

$$+ \sum_{n=1}^{\infty} \frac{a_n - b_n \frac{1}{i}}{2} e^{-inx}$$ ← rewrite as $n = -\infty$

$$\sum_{n=-\infty}^{-1} \frac{a_n - \frac{b_n}{i}}{2} e^{inx}$$

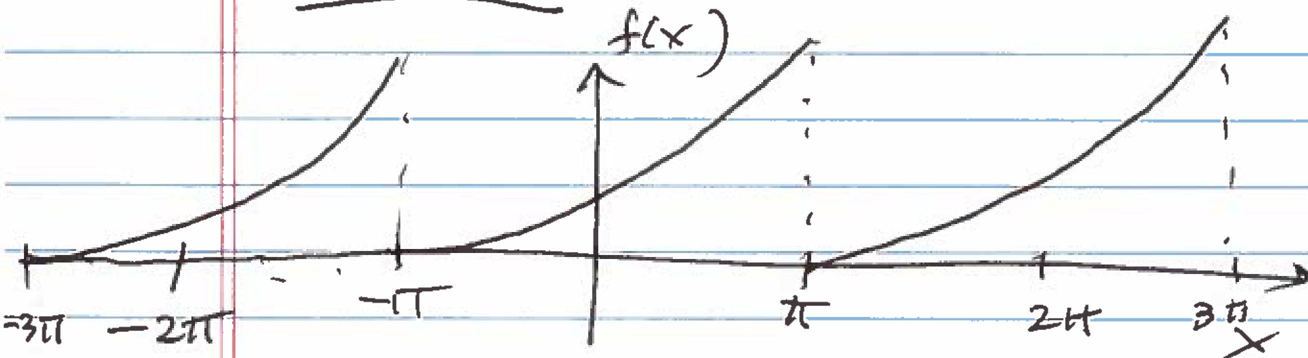
if $f(x)$ was real, a_n, b_n real

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \overbrace{\frac{a_n + \frac{1}{i} b_n}{2}}^{c_n} e^{inx} + \sum_{n=-\infty}^{-1} \underbrace{\frac{a_n - \frac{1}{i} b_n}{2}}_{c_{-n}} e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}, \text{ where } c_{-n} = \overline{c_n}$$

33. 7.8.12

$f(x) = e^x, \; -\pi < x < \pi$



a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx) + \sum_{n=1}^{\infty} (b_n \sin nx)$

(i) $\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} e^x\,dx = e^x \Big|_{-\pi}^{\pi} = e^{\pi} - e^{-\pi}$

$= 2\sinh\pi$

(ii) $\int_{-\pi}^{\pi} f(x) \cos n'x\,dx = \int_{-\infty}^{\infty} e^x \frac{e^{in'x} + e^{-in'x}}{2}\,dx$

$= \frac{e^{x+in'x}}{2(1+in')}\Big|_{-\pi}^{\pi} + \frac{e^{x-in'x}}{2(1-in')}\Big|_{-\pi}^{\pi}$

$= \frac{e^{\pi}(-1)^{n'}}{2(1+in')} - \frac{e^{-\pi}(-1)^{n'}}{2(1+in')}$

$+ \frac{e^{\pi}(-1)^{n'}}{2(1-in')} - \frac{e^{-\pi}(-1)^{n'}}{2(1-in')}$

$= \frac{(-1)^{n'}}{2}\left[\left(\frac{e^{\pi} - e^{-\pi}}{1+in'}\right) + \left(\frac{e^{\pi} - e^{-\pi}}{1-in'}\right)\right]$

$= (-1)^{n'} \sinh\pi \left[\frac{1-in' + 1+in'}{1+n'^2}\right] = (-1)^{n'} \frac{2\sinh\pi}{1+n'^2}$

(iii) $\int_{-\pi}^{\pi} f(x)\sin n'x\,dx = \int_{-\pi}^{\pi} e^{x}\left(\frac{e^{in'x}-e^{-in'x}}{2i}\right)dx$

$$= \frac{e^{x+in'x}}{2i(1+in')}\Bigg|_{-\pi}^{\pi} - \frac{e^{x-in'x}}{2i(1-in')}\Bigg|_{-\pi}^{\pi}$$

$$= \frac{e^{\pi}(-1)^{n'}}{2i(1+in')} - \frac{e^{-\pi}(-1)^{n'}}{2i(1+in')} - \frac{e^{\pi}(-1)^{n'}}{2i(1-in')} + \frac{e^{-\pi}(-1)^{n'}}{2i(1-in')}$$

$$= \frac{(-1)^{n'}}{2i}\left(e^{\pi}-e^{-\pi}\right)\left[\frac{1}{\cancel{2i}(1+in')} - \frac{1}{1-in'}\right]$$

$$= \frac{(-1)^{n'}}{i}\sinh\pi\left[\frac{1-in'-1-in'}{1+n'^2}\right]$$

$$= -\frac{(-1)^{n'}}{1+n'^2}2n'\sinh\pi$$

$$\Rightarrow \quad a_0 = \frac{2}{\pi}\sinh\pi$$

$$a_n = \frac{2}{\pi}\sinh\pi\,\frac{(-1)^n}{1+n^2}$$

$$b_n = -\frac{2}{\pi}\sinh\pi\,\frac{(-1)^n}{1+n^2}$$

$$\Rightarrow \boxed{f(x) = \frac{\sinh\pi}{\pi}\left[\sum_{n=1}^{\infty}\frac{2(-1)^n}{1+n^2}\cos nx + \sum_{n=1}^{\infty}\frac{2(-1)^n}{1+n^2}\sin nx + 1\right]}$$

b) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$\int_{-\pi}^{\pi} f(x) e^{-in'x} dx = 2\pi c_n \delta_{nn'}$$

$$\int_{-\pi}^{\pi} e^{+x-in'x} dx =$$

$$\rightarrow \frac{e^{x-in'x}}{1-in'}\Big|_{-\pi}^{\pi} = \frac{e^{\pi}(-1)^{n'}}{1-in'} - \frac{e^{-\pi}(-1)^{n'}}{1-in'}$$

$$= \frac{(-1)^{n'}}{1-in'}\left(e^{\pi} - e^{-\pi}\right)$$

$$= \frac{2}{1-in'}(-1)^{n'} \sinh\pi$$

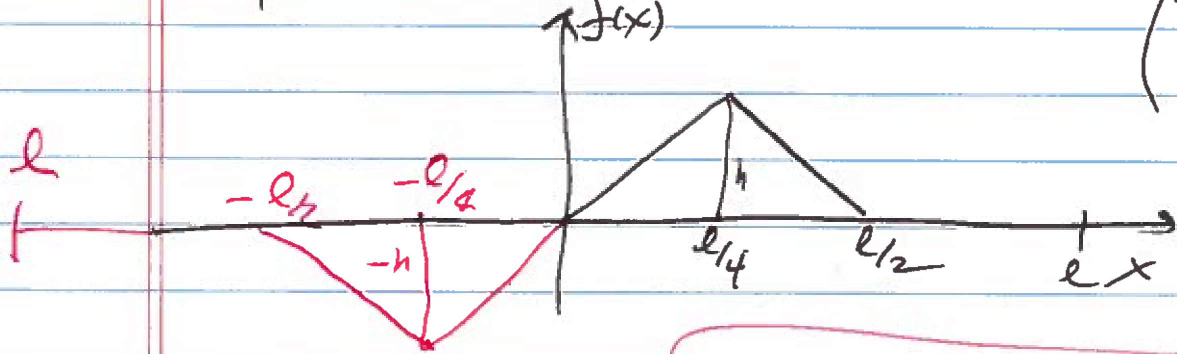
$$\Rightarrow \boxed{f(x) = \sum_{n=-\infty}^{\infty} \frac{\cancel{2}\,1}{1-in'}(-1)^{n'} \frac{\sinh\pi}{\pi} e^{inx}}$$

$$= \frac{\sinh\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n'}}{1-in'} e^{inx}$$

34 7.10.24

Expand as a sine series

$$f(x) = \begin{cases} x\frac{4h}{\ell}, & 0 < x < \frac{\ell}{4} \\ 2h - x\frac{4h}{\ell}, & \frac{\ell}{4} < x < \frac{\ell}{2} \\ 0, & \frac{\ell}{2} < x \end{cases}$$



$\Rightarrow$ Period $= 2\ell \rightarrow$ $\boxed{\text{frequency} = \frac{2\pi n}{2\ell} = \frac{\pi}{\ell} n}$

$$\Rightarrow b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi}{\ell}\right)dx$$

$$= \frac{2}{\ell}\left[\int_0^{\ell/4} x\frac{4h}{\ell}\sin\left(\frac{n\pi}{\ell}x\right)dx + \int_{\ell/4}^{\ell/2}\left(2h - x\frac{4h}{\ell}\right)\sin\left(\frac{n\pi}{\ell}x\right)dx\right.$$

$$= \frac{2}{\ell}\left[\frac{4h}{\ell}\frac{\ell}{n\pi}\frac{\ell}{n\pi}\int_0^{\frac{n\pi}{4}} \xi \sin\xi \, d\xi + \int_{\frac{\ell}{4}}^{\frac{\ell}{2}} 2h\sin\left(\frac{n\pi}{\ell}x\right)dx \quad - \int_{\frac{n\pi}{4}}^{\frac{n\pi}{2}} \frac{4h}{\ell}\frac{\ell^2}{n^2\pi^2}\xi\sin\xi\, d\xi\right.$$

$$= \frac{2}{\ell}\left[\frac{4h}{\ell}\frac{\ell^2}{n^2\pi^2}\left(\sin\xi - \xi\cos\xi\right)\Big|_0^{\frac{n\pi}{4}} - \frac{4h}{\ell}\frac{\ell^2}{n^2\pi^2}\left(\sin\xi - \xi\cos\xi\right)\Big|_{\frac{n\pi}{4}}^{\frac{n\pi}{2}}\right.$$

$$\left. + \frac{2h}{\frac{n\pi}{\ell}}\left(-\cos\left(\frac{n\pi}{\ell}x\right)\right)\Big|_{\ell/4}^{\ell/2}\right]$$

$$b_n = \frac{2}{l}\left[\frac{4hl}{n^2\pi^2}\left\{\left(\sin\frac{n\pi}{4} - \frac{n\pi}{4}\cos\frac{n\pi}{4}\right) - \left(\sin\frac{n\pi}{2} - \sin\frac{n\pi}{4} - \frac{n\pi}{2}\cos\frac{n\pi}{2} + \frac{n\pi}{4}\cos\frac{n\pi}{4}\right)\right\}\right.$$

$$\left. \frac{2hl}{n\pi}\left(\cos\frac{n\pi}{2} - \cos\frac{n\pi}{4}\right)\right]$$

$$b_n = \frac{2}{l}\left[\frac{4hl}{n^2\pi^2}\left\{2\sin\frac{n\pi}{4} - \sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{4} + \frac{n\pi}{2}\cos\frac{n\pi}{2}\right\}\right.$$

$$\left. - \frac{2hl}{n\pi}\left(\cos\frac{n\pi}{2} - \cos\frac{n\pi}{4}\right)\right]$$

$$b_n = \left[\frac{8h}{n^2\pi^2}\left(2\sin\frac{n\pi}{4} - \sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{4} + \frac{n\pi}{2}\cos\frac{n\pi}{2}\right)\right.$$

$$\left. - \frac{4h}{n\pi}\left(\cos\frac{n\pi}{2} - \cos\frac{n\pi}{4}\right)\right]$$

$$= \frac{8h}{(n\pi)^2}\left[2\sin\frac{n\pi}{4} - \sin\frac{n\pi}{2}\right] + \frac{8h}{(n\pi)}\left[\cancel{-\frac{1}{2}\cos\frac{n\pi}{4}} + \cancel{\frac{1}{2}\cos\frac{n\pi}{2}} - \cancel{\frac{1}{2}\cos\frac{n\pi}{2}} + \cancel{\frac{1}{2}\cos\frac{n\pi}{4}}\right.$$

$$= \frac{8h}{(n\pi)^2}\left[2\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\right]$$

first few terms

$$b_1 = \frac{8h}{\pi^2}\left(2\sin\frac{\pi}{4} - 1\right) = \frac{16h}{\pi^2}\left(\sin\frac{\pi}{4} - \frac{1}{2}\right)$$

$$b_2 = \frac{8h}{4\pi^2}(2) = \frac{4h}{\pi^2}$$

$$b_3 = \frac{8h}{9\pi^2}\left(3\sin\frac{3\pi}{4} + 1\right) = \frac{+8h}{3\pi^2}\sin\frac{\pi}{4} + 1$$

$$b_4 = \frac{8h}{16\pi^2}(0) = 0$$

$$b_5 = \frac{8h}{25\pi^2}\left(2\sin\frac{5\pi}{4} - 1\right) = \frac{-16}{25\pi^2}\sin\frac{\pi}{4} - \frac{8h}{25\pi^2}$$

$$b_6 = \frac{8h}{36\pi^2}(-2) = \frac{-4h}{9\pi^2}$$

$$b_7 = \frac{8h}{49\pi^2}\left(2\sin\frac{7\pi}{4} + 1\right) = \frac{-16h}{49\pi^2}\sin\frac{\pi}{4} + \frac{8h}{49\pi^2}$$

$$b_8 = \frac{8h}{64\pi^2}(0) = 0$$

and, in general,

$$f(x) = \sum_{n=1}^{\infty} \frac{16h}{\pi^2}\left(\frac{\sin\left(\frac{n\pi}{4}\right)}{n^2} - \frac{\sin\left(\frac{n\pi}{2}\right)}{2n^2}\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

35 7.11.7

Sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using Prob 5.8

Given $f(x) = 1+x$, find

$\rightarrow f(x) = 1 + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right)$

a) $\int_{-\pi}^{\pi} f(x) f(x)\, dx = \int_{-\pi}^{\pi} (1+x)^2 dx = \int_{-\pi}^{\pi} (1+2x+x^2)\, dx$

$= \left(x + x^2 + \frac{x^3}{3}\right)\Big|_{-\pi}^{\pi}$

$= (2\pi) + (0) + \frac{2\pi^3}{3}$

$\rightarrow \overline{|f(x)|}^2 = 1 + \frac{\pi^2}{3}$

b) $b_n = \frac{2}{n}(-1)^{n-1}, \; a_0 = 2$

c) Parseval's thm

$$\overline{|f(x)|}^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$= 1 + \sum_{n=1}^{\infty}\left(\frac{2}{n^2}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \left[\overline{|f(x)|}^2 - 1\right]\frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\left(1 + \frac{\pi^2}{3} - 1\right)$$

$$= \frac{\pi^2}{6}$$