Homework 1

Due: 6 October 2016, by end of the day

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1. Page 106, 3.4, Problem 24
2. Page 113, 3.5, Problem 45
3. Page 284, 6.3, Problem 10
4. Page 284, 6.3, Problem 12
5. Page 284, 6.3, Problem 14
6. Page 285, 6.3, Problem 18
7. page 289, 6.4, Problem 6
8. page 289, 6.4, Problem 10

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9. Using vector methods prove the following trigonmetric relations

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

10. Show that the dot product behaves as a scalar under rotation.

HW #1 3.4(24), 3.5(45), 6.3(10), 6.3(12), 6.3(14), 6.3(18), 6.4(6), 6.4(10); Show $\cos(x+y)$, $\sin(x,y)$, $\vec{A}\cdot\vec{B}$ is a scalar

#1 3.4(24)

What is $(\vec{A}\times\vec{B}) + (\vec{A}\cdot\vec{B})^2$?

$$= (A_yB_z - A_zB_y,\ A_zB_x - A_xB_z,\ A_xB_y - A_yB_x)^2$$
$$+ \left[A_xB_x + A_yB_y + B_zA_z\right]^2$$

$$= \left[A_y^2B_z^2 - 2A_yB_zA_zB_y + A_z^2B_y^2 + A_z^2B_x^2 - 2A_zB_xA_xB_z + A_x^2B_z^2 + A_x^2B_y^2 - 2A_xB_yA_yB_x + A_y^2B_x^2\right]$$
$$+ \left[A_x^2B_x^2 + 2A_xB_xA_yB_y + 2A_xB_xB_zA_z + A_y^2B_y^2 + 2A_yB_yB_zA_z + B_z^2A_z^2\right]$$

cancel cross-terms

$$= A_y^2(B_z^2 + B_x^2) + A_z^2(B_y^2 + B_x^2) + A_x^2(B_z^2 + B_y^2)$$
$$= A_y^2(B_z^2 + B_x^2 + B_y^2) + A_z^2(B_y^2 + B_x^2 + B_z^2) + A_x^2(B_z^2 + B_y^2 + B_x^2)$$
$$- (A_y^2B_y^2 + B_z^2A_z^2 + A_x^2B_x^2)$$

$$\boxed{= (\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B}) - (\vec{A}\cdot\vec{B})}$$

#2 3,5(95)

a particle travels along line $\left(\frac{x-3}{2}\right)=\left(\frac{y+1}{-2}\right)=\left(\frac{z-1}{1}\right)$.

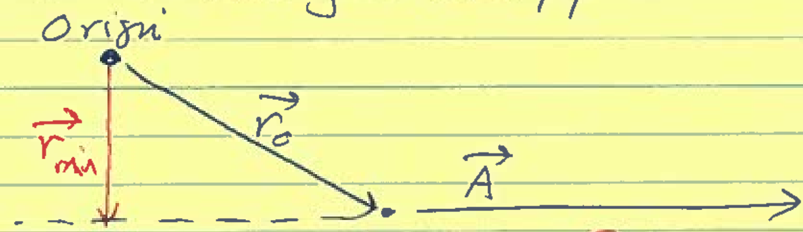
(a) Write the equation in form $\vec{r}=\vec{r_0}+\vec{A}t$

$\vec{r}=\vec{r}_0+\vec{A}t \Rightarrow (\vec{r}-\vec{r}_0)=\vec{A}t$, where $\vec{A}=(a,b,c)$

$\Rightarrow \vec{r}_0=(3,-1,1)$ & $\vec{A}=(2,-2,1)$

$$\Rightarrow \boxed{\vec{r}=(3,-1,1)+(2,-2,1)t}$$

(b) find the distance of closest approach to the origin



the distance of closest approach occurs when $\vec{r}_{min}\perp\vec{A}$

$$\Rightarrow \left[(3,-1,1)+(2,-2,1)t\right]\cdot(2,-2,1)=0$$

$$(9+9t)=0 \Rightarrow t=-1$$

and $\vec{r}_{min}=(3,-1,1)+(-2,2,-1)=(1,1,0)$

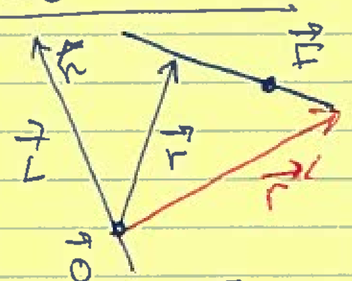
$$\Rightarrow \boxed{|\vec{r}_{min}|=\sqrt{1+1}=\sqrt{2}}$$

(c) okay, $\vec{r}\cdot\vec{A}=0 \Rightarrow \vec{r}_0\cdot\vec{A}+\vec{A}\cdot\vec{A}\,t=0$

$$\Rightarrow \boxed{t=\frac{-\vec{r}_0\cdot\vec{A}}{|\vec{A}|^2}}$$

and $t_{min}=-\frac{(3,-1,1)\cdot(2,-2,1)}{|(2,-2,1)|^2}=-\frac{9}{9}=-1$ ✓

#36.3 (10)



3.5

(a) Show that $\vec r' \times \vec F = \vec r \times \vec F$

Subtract $\vec r'$ from $\vec r$ forms

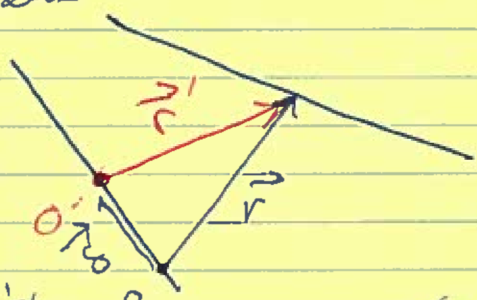
$\Rightarrow \vec r \times \vec F - \vec r' \times \vec F$

$(\vec r - \vec r') \times \vec F$

since $\vec r, \vec r'$ start from the same point and end on $\vec F$ $\Rightarrow \vec \ell = \vec r - \vec r'$ is parallel (or anti-parallel) to $\vec F$ $\Rightarrow \vec \ell \times \vec F = 0$

and $\vec r \times \vec F = \vec r' \times \vec F$

(b) Consider



Show that this shift does not change

$\hat n \cdot (\vec r \times \vec F)$

Let the "origin" shift be given by $\vec \delta$ $\longrightarrow$ $\hat n \cdot (\vec r' \times \vec F) = \hat n \cdot (\vec r \times \vec F) = ?$

$\hat n \cdot [(\vec r - \vec \delta) \times \vec F] = \hat n \cdot [(\vec r \times \vec F)]$

note: $\vec \delta \times \vec F$ is $\perp$ to $\vec \delta$, $\vec F$

$\Rightarrow -\hat n \cdot (\vec \delta \times \vec F) = 0$

and

$\Rightarrow \hat n \cdot (\vec r' \times \vec F) - \hat n \cdot (\vec r \times \vec F) =$

$\Rightarrow \hat n \cdot (\vec r \times \vec F)$ is unchanged

#4 6.3 (12)

Simplify

a) $(\vec{A}\cdot\vec{B})^2 - [\underbrace{(\vec{A}\times\vec{B})\times\vec{B}}]\cdot\vec{A}$ using BAC−CAB

$-\vec{B}\times(\vec{A}\times\vec{B})$

$-[\vec{A}(\vec{B}\cdot\vec{B}) - \vec{B}(\vec{A}\cdot\vec{B})]$

$\cancel{2(\vec{A}\cdot\vec{B})^2} -$

$(\vec{A}\cdot\vec{B})^2 - [-\vec{A}(\vec{B}\cdot\vec{B}) + \vec{B}(\vec{A}\cdot\vec{B})]\cdot\vec{A}$

$= (\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B})$

b) Prove

$\underbrace{(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})}_{\text{interchange } \cdot \text{ \& } \times} = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$

$[(A\times B)\times\vec{C}]\cdot\vec{D}$

$= [-\vec{C}\times(\vec{A}\times\vec{B})]\cdot\vec{D}$

$= -[\vec{A}(\vec{C}\cdot\vec{B}) - \vec{B}(\vec{A}\cdot\vec{C})]\cdot\vec{D}$

$= -(\vec{A}\cdot\vec{D})(\vec{C}\cdot\vec{B}) + (\vec{B}\cdot\vec{D})(\vec{A}\cdot\vec{C})$

~~#5~~ 6.3(14)

Prove $\vec{A}\times(\vec{B}\times\vec{C})+\vec{B}\times(\vec{C}\times\vec{A})+\vec{C}\times(\vec{A}\times\vec{B})=0$

$$\left[\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})\right]+\left[\vec{C}(\vec{A}\cdot\vec{B})-\vec{A}(\vec{B}\cdot\vec{C})\right]+\left[\vec{A}(\vec{B}\cdot\vec{C})-\vec{B}(\vec{A}\cdot\vec{C})\right]$$

$$=0$$

#6 6.3(17)

$$\vec{F}_1 \propto \vec{V}_1\times(\vec{V}_2\times\vec{r})$$
$$\vec{F}_2 \propto \vec{V}_2\times(\vec{V}_1\times(-\vec{r}))$$

Add proportionality constants f_0, f_1 and then add $\vec{F}_1$ & $\vec{F}_2$

$$f_0\vec{V}_1\times(\vec{V}_2\times\vec{r})+f_1\vec{V}_2\times(\vec{V}_1\times[-\vec{r}])$$

$$f_0\left[\vec{V}_2(\vec{V}_1\cdot\vec{r})-\vec{r}(\vec{V}_1\cdot\vec{V}_2)\right]+f_1\left[\vec{V}_1(-\vec{V}_2\cdot\vec{r})+\vec{r}(\vec{V}_1\cdot\vec{V}_2)\right]$$

$$=f_0\vec{V}_2(\vec{V}_1\cdot\vec{r})+f_1\vec{V}_1(-\vec{V}_2\cdot\vec{r})$$

if $f_0=f_1 \Rightarrow$ if $\vec{r}\times(\vec{V}_1\times\vec{V}_2)=0 \rightarrow$ sum of forces $=0$

26.4 (6)

$\vec{f} = q\vec{v}\times\vec{B}$ $\quad \perp \vec{v}$ & $\vec{B}$

(a) show $\vec{f} \perp \vec{v}$

$$\vec{v}\cdot\vec{f} = q\vec{v}\times(\vec{v}\times\vec{B}) \Rightarrow q\vec{v}\times(\vec{v}\times\vec{B}) = 0$$

(b) show $\vec{v}\cdot\vec{v}$ & $\vec{f}\cdot\vec{f}$ are constants

$$\vec{v}\cdot\vec{f} = m\vec{v}\cdot\frac{d}{dt}\vec{v} = \frac{m}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) = 0$$

$\Rightarrow (\vec{v}\cdot\vec{v})$ is constant (as asked)

c) take $\vec{f}\cdot\vec{f} = (q\vec{v}\times\vec{B})\cdot(q\vec{v}\times\vec{B})$

$$\cancel{= q^2\Big[\vec{v}\big(\vec{B}\cdot\underbrace{[\vec{v}\times\vec{B}]}_{\perp\,\vec{v}\,\&\,\vec{B}}\big) - \vec{B}\big(\vec{v}\cdot\underbrace{[\vec{v}\times\vec{B}]}_{\perp\,\vec{v}\,\&\,\vec{B}}\big)\Big]}$$

$$\cancel{= 0}$$

$$= (q\vec{v}\times(\vec{B}\times q\vec{v}))\cdot\vec{B}$$

$$= q^2\left[\vec{B}(\vec{v}\cdot\vec{v}) - \vec{v}(\vec{v}\cdot\vec{B})\right]\cdot\vec{B}$$

$$= \underbrace{q^2|\vec{B}|^2|\vec{v}|^2}_{\text{constant}} - q^2|\vec{v}\cdot\vec{B}|^2$$

b/c $\vec{v}$ is in (x-y) plane & $\vec{B}$ is in z-direction

$\rightarrow |\vec{v}\cdot\vec{B}| = 0$

$\rightarrow \boxed{\vec{f}\cdot\vec{f} = \text{constant}}$

#8 6.4 (10)

if $\vec{v}(t)$ is a vector function of t, evaluate

$$\int \left(\vec{v} \times \frac{d^2\vec{v}}{dt^2} \right) dt$$

Construct

$$\frac{d}{dt}\left(\vec{v} \times \frac{d\vec{v}}{dt} \right) = \cancelto{0}{\frac{d\vec{v}}{dt} \times \frac{d\vec{v}}{dt}} + \vec{v} \times \frac{d^2\vec{v}}{dt^2}$$

$$\Rightarrow \int \left(\vec{v} \times \frac{d^2\vec{v}}{dt^2} \right) dt$$

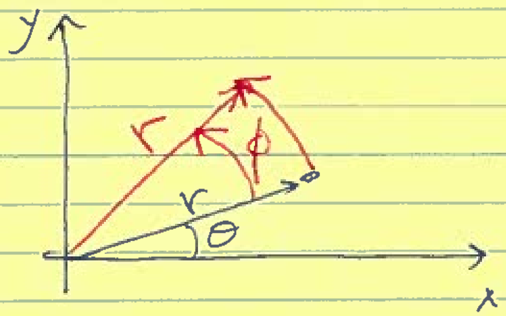
$$= \int \frac{d}{dt}\left[\vec{v} \times \frac{d\vec{v}}{dt} \right] dt$$

$$= \int d\left(\vec{v} \times \frac{d\vec{v}}{dt} \right)$$

constant vector

$$\rightarrow \boxed{\int \left(\vec{v} \times \frac{d^2\vec{v}}{dt^2} \right) dt = \vec{v} \times \frac{d\vec{v}}{dt} + \vec{A}}$$

#9 2∢ cos & sin formulas



rotate $\vec{r}$ by ∢ ϕ about z-axis

$$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{active rotation}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

note: $x' = r\cos(\theta+\phi)$
and
$y' = r\sin(\theta+\phi)$

$$\Rightarrow \left.\begin{aligned} r\cos(\theta+\phi) &= x\cos\phi - y\sin\phi \\ r\sin(\theta+\phi) &= x\sin\phi + y\cos\phi \end{aligned}\right) \text{A}$$

note: $$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right) \text{B}$$

A & B $\Rightarrow$ $$\boxed{\begin{aligned} \cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin(\theta+\phi) &= \cos\theta\sin\phi + \sin\theta\cos\phi \end{aligned}}$$

10 Show $\vec{A}\cdot\vec{B}$ is a scalar

Show that $\vec{A}'\cdot\vec{B}' = \vec{A}\cdot\vec{B}$

write out $\vec{A}'\cdot\vec{B}'$ using rotations to see if we get the "correct" answer

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}\cdot\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$$

$$= \begin{pmatrix} A_x\cos\theta + A_y\sin\theta \\ -A_x\sin\theta + A_y\cos\theta \\ A_z \end{pmatrix}\cdot\begin{pmatrix} B_x\cos\theta + B_y\sin\theta \\ -B_x\sin\theta + B_y\cos\theta \\ B_z \end{pmatrix}$$

actually should write this as $\begin{pmatrix} A_x\cos\theta + A_y\sin\theta & -A_x\sin\theta + A_y\cos\theta & A_z \end{pmatrix}$

$$\begin{aligned} &= (A_x\cos\theta + A_y\sin\theta)(B_x\cos\theta + B_y\sin\theta) \\ &\quad + (-A_x\sin\theta + A_y\cos\theta)(-B_x\sin\theta + B_y\cos\theta) \\ &\quad + A_z B_z \\ &= (A_xB_x\cos^2\theta + A_xB_y\cos\theta\sin\theta + A_yB_x\sin\theta\cos\theta + A_yB_y\sin^2\theta) \\ &\quad + (A_xB_x\sin^2\theta - A_xB_y\sin\theta\cos\theta - A_yB_x\cos\theta\sin\theta + A_yB_y\cos^2\theta) \\ &\quad + A_zB_z \\ &= A_xB_x + A_yB_y + A_zB_z \\ &= \vec{A}\cdot\vec{B} \quad \checkmark \end{aligned}$$