Homework 8

The problems are relevant for the Final; they are not to be turned in.

12.4.3, 12.5.14, 12.6.16, 12.11.11, 13.10.12, 13.10.21, 13.10.20

Hw #8

12.4.3

Find P_0, P_1, P_2, P_3, P_4 from Rodrigues' formula.

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

(i) $P_0(x) = \frac{1}{2^0 0!}\frac{d^0}{dx^0}(x^2-1)^0 = \boxed{1 = P_0(x)}$

(ii) $P_1(x) = \frac{1}{2^1 1!}\frac{d}{dx}(x^2-1) = \frac{2x}{2} = \boxed{x = P_1(x)}$

(iii) $P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{4.2}\frac{d}{dx}(2(x^2-1)2x)$

$$= \frac{1}{4.2}\cdot 4\left[(x^2-1) + x(2x)\right]$$

$$\boxed{P_2(x) = \frac{1}{2}\left[3x^2-1\right]}$$

(iv) $P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^2}(x^2-1)^3 = \frac{1}{2^3 3!}\left[\frac{d^2}{dx^2}\left(3[x^2-1]^2 2x\right)\right]$

$$= \frac{1}{2^3 3!}\frac{d}{dx}\left[6(x^2-1)^2 + 12(x^2-1)4x^2\right]$$

$$= \frac{1}{2^3 3!}\left[24(x^2-1)x + 48x(x^2-1) + 48(x^2 2x)\right]$$

$$= \frac{1}{8\cdot 6}\left[168x^3 - 72x\right]$$

$$= \frac{1}{6}\left[21x^3 - 9x\right]$$

$$= \frac{1}{2}\left[7x^3 - 3x\right]$$

$$= \frac{1}{2^3 3!}\left[\frac{d^2}{dx^2}\left\{3(x^2-1)^2 2x\right\}\right]$$

$$= \frac{6}{2^3 3!}\frac{d}{dx}\left\{(x^2-1)^2 + 2(x^2-1)2x\,x\right\}$$

$$= \frac{6}{2^3 3!}\frac{d}{dx}\left\{(x^2-1)^2 + 4(x^4-x^2)\right\}$$

$$= \frac{6}{2^3 3!}\left\{2(x^2-1)2x + 16x^3 - 8x\right\}$$

$$= \frac{\cancel{6}}{2\,\cancel{2^3}\,\cancel{3!}}\left\{x^3\left[\frac{4+16}{4}\right] + x\left[\frac{-4-8}{4}\right]\right\}$$

$$\boxed{P_3(x) = \frac{1}{2}\left\{5x^3 - 3x\right\}}$$

12.5.14

a polynomial, $a\mu^n + b\mu^{n-1} + c\mu^{n-2} \cdots + d\mu = 0$
can be written as a linear combination of Legendre polynomials w/order $l \leq n$.

a) From Rodrigues' formula
$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(\mu^2 - 1)^l$$
$$= \frac{1}{2^l l!}\frac{d^l}{dx^l}(-1)^l\left[(1-\mu^2)^l\right]$$
$$= \frac{(-1)^l}{2^l l!}\frac{d^l}{dx^l}\left[1 - l\mu^2 + \frac{l(l-1)}{2!}(-\mu^2)^2 + \frac{l(l-1)(l-2)}{3!}(-\mu^2)^3 + \cdots\right]$$

polynomial of order $2l$

$\Rightarrow P_l(x)$ will be a polynomial of order $\frac{d^l}{dx^l}x^{2l} \approx x^l$

b.) So, consider x^3

(ii) $P_3 = \frac{1}{2}(5x^3 - 3x) \rightarrow x^3 = \frac{(2P_3 + 3x)}{5}$
$$= \left(\frac{2P_3 + 3P_1}{5}\right)$$

(iii) Similarly, we can construct ~~any~~ series of Legendre polynomials to describe an arbitrary polynomial of order n using P_l where $l \leq n$.

12-6-16 a mystery to me at this point

12-11-11

Solve $36x^2y'' + (5-9x^2)y = 0$ by series method (Frobenius)

okay, let $y = \sum_{n=0}^{\infty} a_n x^{n+k}$

$$\rightarrow 36x^2 \sum_{n=0}^{\infty}\left(a_n(n+k)(n+k-1)x^{n+k-2}\right) + (5-9x^2)\sum_{n=0}^{\infty} a_n x^{n+k} = 0$$

~~find terms w/ smallest powers of x~~

$$36\sum_{n=0}^{\infty} a_n(n+k)(n+k-1)x^{n+k} + 5\sum_{n=0}^{\infty} a_n x^{n+k} - 9\sum_{n=0}^{\infty} a_n x^{n+k+2} = 0$$

look at case when $n=0$ & lowest power of x^{α} terms

(i) $x^{k}(36a_0 k(k-1) + 5a_0)$ must sum to 0, in general

$$\Rightarrow a_0[5 + 36k(k-1)] = 0$$

$\hookrightarrow a_0 = 0$ or $k^2 - k + \frac{5}{36} = 0$

$$k = \frac{1 \pm \sqrt{1 - \frac{5}{9}}}{2}$$

$$k = \frac{1}{2} \pm \frac{1}{3} = \frac{5}{6}, \frac{1}{6}$$

(ii) $x^{k+1}(36a_1(k+1)k + 5a_1) = 0$

$\hookrightarrow a_1[5 + 36k(k+1)] = 0$, $a_1 = 0$ or

$36k^2 + 36k + 5 = 0$, $k^2 + k + \frac{5}{36} = 0$

$$k = \frac{-1 \pm \sqrt{1 - \frac{20}{36}}}{2} = -\frac{1}{6}, -\frac{5}{6}$$

form recurrence relation for a_n

$$\rightarrow 36\sum_{n=0}^{\infty} a_{n+2}(n+k+2)(n+k+1)x^{n+k+2} + 5\sum_{n=0}^{\infty} a_n x^{n+k} - 9\sum_{n=0}^{\infty} a_n x^{n+k+2}$$

b/c $n=0,1$ terms do not survive
$y'' \rightarrow$ series actually starts w/ 2.
But re-define series to start at $n=0$, but change $n \rightarrow n+2$ in series

(i) now, gather terms w/ powers x^{n+k+2},

$$\left(36a_{n+2}(n+k+2)(n+k+1) + 5a_{n+2} - 9a_n\right)x^{n+k+2}$$

↑ again, ( ) term → 0, in general

$$a_{n+2}\left[36(n+k+2)(n+k+1)+5\right] - 9a_n = 0$$

$$\boxed{\frac{a_{n+2}}{a_n} = \frac{9}{36(n+k+1)(n+k+2)+5}}$$

(ii) for case $K = +1/6 \rightarrow$ $(\rightarrow a_1 = 0)$

$$\frac{a_{n+2}}{a_n} = \frac{9}{36(n+7/6)(n+13/6)+5}$$

$$= \frac{9}{36(n^2+\frac{10}{3}n+\frac{91}{36})+5}$$

$$= \frac{9}{36n^2+120n+91+5}$$

$$= \frac{9}{36n^2+120n+96}$$

$$= \frac{3}{12n^2+40n+32}$$

$$= \frac{3}{4(3n^2+10n+8)}$$

$$\boxed{\frac{a_{n+2}}{a_n} = \frac{3}{4(3n+4)(n+2)}}$$

$$\Rightarrow y_1 = \sum_{\substack{n=0 \\ \text{even}}}^{\infty} a_n x^{n+\frac{1}{6}} = x^{1/6}\left[a_0 + a_2 x^2 + a_4 x^4 + \cdots\right]$$

$$= x^{1/6}\left[a_0 + \frac{3}{32}a_0 x^2 + \frac{9}{5{,}120}a_0 x^4 + \cdots\right]$$

$$\boxed{y_1 = a_0 x^{1/6}\left[1 + \frac{3}{32}x^2 + \frac{9}{5{,}120}x^4 + \cdots\right]}$$

(iii) for case $k=-\frac{1}{6} \rightarrow \frac{a_{n+2}}{a_n} = \frac{9}{36\left(n+\frac{5}{6}\right)\left(n+\frac{11}{6}\right)+5}$

$(\rightarrow a_0 = 0)$

$$= \frac{9}{36\left(n^2+\frac{16}{6}n+\frac{55}{36}\right)+5}$$

$$= \frac{9}{36n^2+96n+55+5}$$

$$= \frac{9}{6\left[6n^2+16n+10\right]}$$

$$= \frac{9}{12\left[3n^2+8n+5\right]}$$

$$\boxed{\frac{a_{n+2}}{a_n} = \frac{3}{4(3n+1)(n+5)}}$$

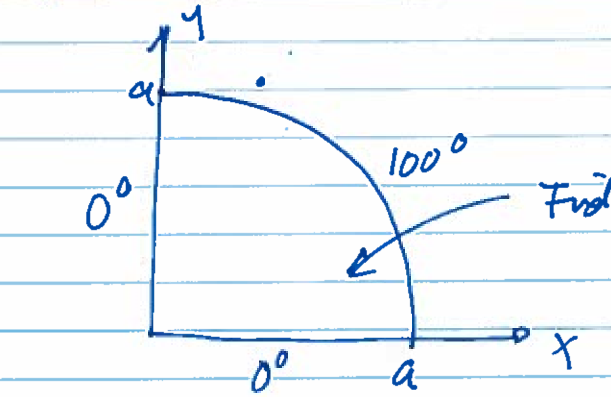
$$\Rightarrow y_2 = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} a_n x^{n-\frac{1}{6}} = x^{-1/6}\left[a_1 + a_3 x^3 + a_5 x^5 + \cdots\right]$$

$$= x^{-1/6}\left[a_1 + \frac{1}{32} a_1 x^3 + \frac{3}{10{,}240} a_1 x^5 + \cdots\right]$$

$$\boxed{y_2 = x^{-1/6} a_1 \left[1 + \frac{1}{32} x^3 + \frac{3}{10{,}240} x^5 + \cdots\right]}$$

other cases add no new information; we have two independent solutions, y_1 and y_2 defined by the arbitrary constants a_0 & a_1

13.10.12



Find $T(r,\theta)$ inside circle

$$\boxed{\nabla^2 T = \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) T = 0}$$

Use cylindrical coordinates (sorry)

let $T(r,\theta) = f(r) g(\theta)$

$$\rightarrow g(\theta)\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\right] f + f \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} g = 0$$

$$\underbrace{\frac{r}{\cancel{r} f}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right)}_{+m^2} + \underbrace{\frac{1}{\cancel{r^2} g}\frac{\partial^2}{\partial \theta^2} g}_{-m^2} = 0$$

$$\longrightarrow g = g_0 \cos m\theta + g_1 \sin m\theta$$

$$\rightarrow r\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) = m^2 f$$

let $f = a r^b \Rightarrow r\frac{\partial}{\partial r}\left[\cancel{r} a b r^{b-\cancel{1}}\right] = m^2 a r^b$

$$\cancel{r} a b (b) r^{b-\cancel{1}} = m^2 a r^b$$

$$a r^b \left[m^2 - \cancel{a} b^2\right] = 0$$

$$\rightarrow m = \pm b$$

$\Rightarrow$ a general solution is $\boxed{T(r,\theta) = \left(a_0 r^b + \frac{a_1}{r^b}\right)\left(g_0 \cos m\theta + g_1 \sin m\theta\right)}$

at $\theta = 0, \frac{\pi}{2}, T \to 0 \Rightarrow g_0 = 0, m\frac{\pi}{2} = n\pi$

$$m = \frac{2n}{\cancel{4}}$$

$$\Rightarrow T(r,\theta) = \left(a_0 r^b + \frac{a_1}{r^b}\right) g_n \sin\left(\frac{2n}{\cancel{\pi}}\theta\right)$$

we want $T(r,\theta)$ well-behaved at $r=0 \to a_1 = 0$

$$\Rightarrow \boxed{T(r,\theta) = c_n r^{2n} \sin(2n\theta)}$$

at $r = a$, $T(a,\theta) = 100$

$$\Rightarrow 100 = \sum_{n=1}^{\infty} C_n a^{2n} \sin(2n\theta)$$

find C_n

$$\int_0^{\pi/2} 100 \sin(2n'\theta)\, d\theta = \sum_{n=1}^{\infty} C_n a^{2n} \int_0^{\pi/2} \sin(2n\theta) \sin(2n'\theta)\, d\theta$$

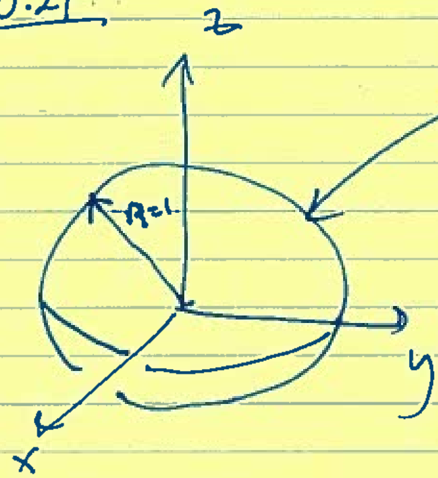
$$-\frac{100}{2n'} \cos(2n'\theta)\Big|_0^{\frac{\pi}{2}} = \sum_{n=1}^{\infty} C_n a^{2n} \int_0^{\pi/2} \sin(2n\theta)\sin(2n'\theta)\, d\theta$$

$$\to \frac{200}{2n}, n \text{ odd} \quad = \frac{\pi}{4} a^{2n} C_n \delta_{nn'}$$

$$\to C_n = \frac{400}{\pi n a^{2n}}$$

and $$\boxed{T(r,\theta) = \sum_{n=1,\ \text{odd}}^{\infty} \frac{400}{\pi n a^{2n}} r^{2n} \sin(2n\theta)}$$

13.10.21



$u = \sin^2\theta + \cos^3\theta$, find $u(r,\theta)$ for $r<1$

$$u(r,\theta) = \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right)P_l(\theta)$$

for axisymmetric solution

a) BCs is that $u = \sin^2\theta + \cos^3\theta$ at $r=1$

a) express in series of Legendre polynomials

$P_3 = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$ b/c $u = \cos^3\theta + \sin^2\theta$
$= \cos^3\theta + 1 - \cos^2\theta$

and we need up to P_3

(i) $2P_3(\theta) = 5\cos^3\theta - 3\cos\theta \rightarrow \cos^3\theta = \frac{1}{5}(2P_3 + 3\cos\theta)$

$P_1(\theta) = \cos\theta$

$= \frac{1}{5}(2P_3 + 3P_1)$

(ii) $2P_2 = 3\cos^2\theta - 1 \rightarrow \cos^2\theta = \frac{1}{3}(2P_2 - 1) = \frac{1}{3}(2P_2 - P_0)$

$$\Longrightarrow u = \cos^3\theta + \sin^2\theta = \frac{1}{5}(2P_3 + 3P_1) + P_0 - \left(\frac{2P_2 - P_0}{3}\right)$$

$$\boxed{u(r=1) = \frac{2}{5}P_3 - \frac{2}{3}P_2 + \frac{3}{5}P_1 + \frac{2}{3}P_0}$$

b) we don't want to blow up at $r=0 \rightarrow b_\ell = 0$
and so

$$\boxed{u(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\theta)}$$

c) determine a_ℓs from BCs, where

$$u(r=1,\theta) = \frac{2}{5}P_3 - \frac{2}{3}P_2 + \frac{3}{5}P_1 + \frac{2}{3}P_0$$

$$\Rightarrow a_3 = \frac{2}{5},\ a_2 = -\frac{2}{3},\ a_1 = \frac{3}{5},\ a_0 = \frac{2}{3}$$

and all other $a_\ell = 0$

$$\Rightarrow \boxed{u(r,\theta) = \frac{2}{3} + \frac{3}{5} r P_1(\theta) - \frac{2}{3} r^2 P_2(\theta) + \frac{2}{5} r^3 P_3(\theta)}$$