Homework 1

Due: 4 October 2018, by end of the work day

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1. Page 106, 3.4, Problem 25
2. Page 113, 3.5, Problem 45
3. Page 132, 3.7, Problem 31--this was mistakenly given as in 3.8, earlier
4. Page 284, 6.3, Problem 13
5. Page 284, 6.3, Problem 14
6. Page 289, 6.4, Problem 6
7. Page 289, 6.4, Problem 10
8. Show using vector methods the following trigonmetric relations:

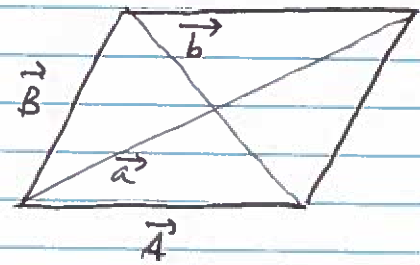
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

HW #1: 3.4.25, 3.5.45, 3.7.31, 6.3.13, 6.3.14, 6.4.6, 6.4.6

Show $\cos(A+B)$ and $\sin(A+B)$ identities

#1 3.4.25

Show for



that

$$a^2+b^2=\left(|\vec{A}|^2+|\vec{B}|^2\right)2$$

Sol^{n}

$$\text{(i) } a^2=\vec{a}\cdot\vec{a}=(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})$$

$$=|\vec{A}|^2+|\vec{B}|^2+\underbrace{\vec{A}\cdot\vec{B}+\vec{B}\cdot\vec{A}}_{2\vec{A}\cdot\vec{B}}$$

, dot product is commutative

$$=|\vec{A}|^2+|\vec{B}|^2+2\vec{A}\cdot\vec{B}$$

$$\text{(ii) } b^2=\vec{b}\cdot\vec{b}=(\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$$

$$=|\vec{A}|^2+|\vec{B}|^2-\vec{A}\cdot\vec{B}-\vec{B}\cdot\vec{A}$$

$$=|\vec{A}|^2+|\vec{B}|^2-2\vec{A}\cdot\vec{B},$$ dot product is commutative

$$\Rightarrow a^2+b^2=2\left[|\vec{A}|^2+|\vec{B}|^2\right]$$

#2 3,5(45)

a particle travels along line $\left(\frac{x-3}{2}\right)=\left(\frac{y+1}{-2}\right)=\left(\frac{z-1}{1}\right)$.

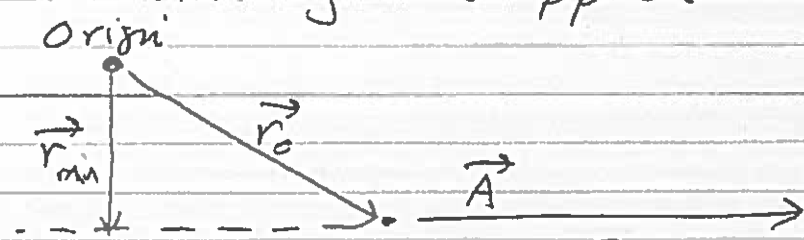
(a) Write the equation in form $\vec{r} = \vec{r}_0 + \vec{A}t$

$\vec{r} = \vec{r}_0 + \vec{A}t \Rightarrow (\vec{r} - \vec{r}_0) = At$, where $\vec{A} = (a, b, c)$

$\Rightarrow \vec{r}_0 = (3,-1,1)$ & $\vec{A} = (2,-2,1)$

$\Rightarrow \boxed{\vec{r} = (3,-1,1)+(2,-2,1)t}$

(b) find the distance of closest approach to the origin



the distance of closest approach occurs when $\vec{r}_{min} \perp \vec{A}$

$\Rightarrow [(3,-1,1)+(2,-2,1)t]\cdot(2,-2,1) = 0$

$(9+9t) = 0 \Rightarrow t = -1$

and $\vec{r}_{min} = (3,-1,1) + (-2,2,-1) = (1,1,0)$

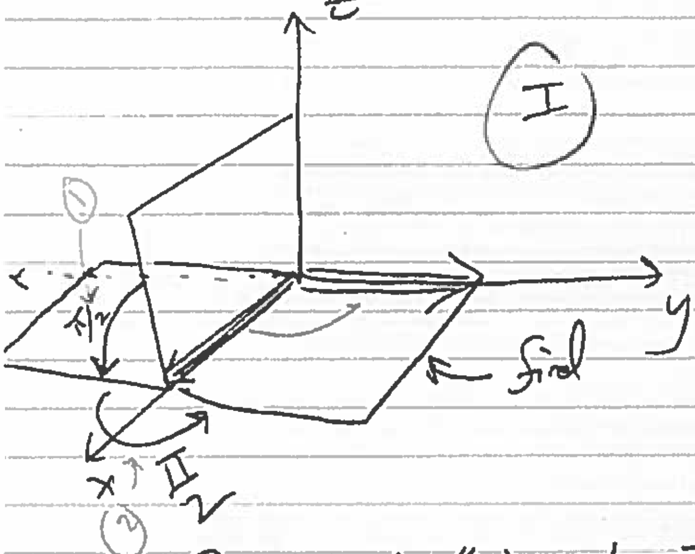
$\Rightarrow \boxed{|\vec{r}_{min}| = \sqrt{1+1} = \sqrt{2}}$

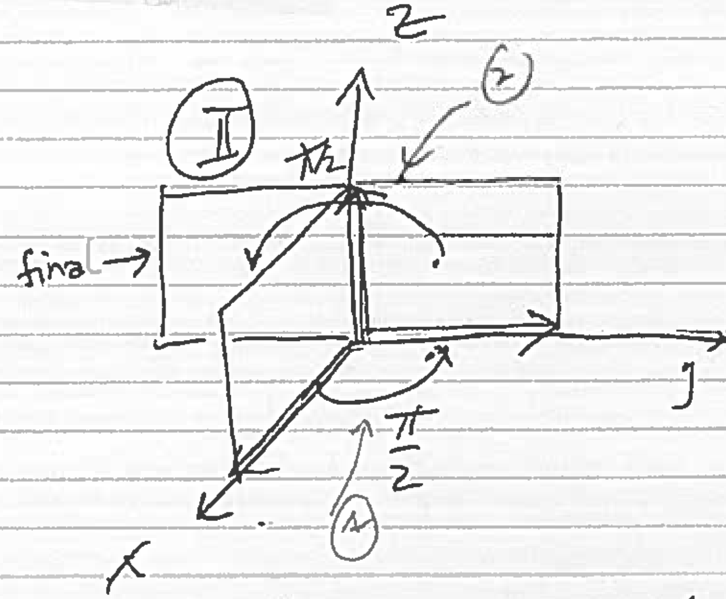
(c) okay, $\vec{r}\cdot\vec{A} = 0 \Rightarrow \vec{r}_0\cdot\vec{A} + \vec{A}\cdot\vec{A}\,t = 0$

$\Rightarrow \boxed{t = \frac{-\vec{r}_0\cdot\vec{A}}{|\vec{A}|^2}}$

and $t_{min} = -\frac{(3,-1,1)\cdot(2,-2,1)}{|(2,-2,1)|^2} = -\frac{9}{9} = 1$ ✓

#3 3.7(31)





I (a)

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ +\sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{about } z\text{-axis by } \pi/2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ 0 & +\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}}_{\text{about } x\text{-axis by } \pi/2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ +1 & 0 & 0 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ +x \\ +y \end{pmatrix}}$$

$\Rightarrow$ z rotates into x & x rotates into y (represents the operation as 1 rotation)

(b) find axis of rotation. Rotate by $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. If we rotate about the vector that points along the axis then the rotation will leave the vector unchanged!

#3 (a)

$$\begin{pmatrix}x''\\y''\\z''\end{pmatrix}=\underbrace{\begin{pmatrix}1&0&0\\0&\cos\frac{\pi}{2}&-\sin\frac{\pi}{2}\\0&\sin\frac{\pi}{2}&\cos\frac{\pi}{2}\end{pmatrix}}_{\text{rotate about } x\text{-axis by } \pi/2}\underbrace{\begin{pmatrix}\cos\frac{\pi}{2}&-\sin\frac{\pi}{2}&0\\\sin\frac{\pi}{2}&\cos\frac{\pi}{2}&0\\0&0&1\end{pmatrix}}_{\text{rotate about } z\text{-axis by } \pi/2}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\boxed{\begin{pmatrix}x''\\y''\\z''\end{pmatrix}=\begin{pmatrix}0&-1&0\\0&0&-1\\1&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-y\\-z\\x\end{pmatrix}}$$

$\Rightarrow$ x rotates into z & z rotates into $-y$

(b)

$$\begin{pmatrix}x_a\\y_a\\z_a\end{pmatrix}=\begin{pmatrix}0&-1&0\\0&0&-1\\1&0&0\end{pmatrix}\begin{pmatrix}x_a\\y_a\\z_a\end{pmatrix}$$

(represents operation as one rotation)

$$=\begin{pmatrix}-y_a\\-z_a\\x_a\end{pmatrix}$$

$\Rightarrow x_a=-y_a,\ y_a=-z_a,\ x_a=z_a$. Set $x_a=1$

$\rightarrow x_a=1,\ y_a=-1,\ z_a=1$ and the rotation axis points in direction

$$\boxed{\frac{1}{\sqrt{3}}(1,-1,1)}$$

$\Rightarrow$ rotates $\frac{1}{3}$ of way around $\Rightarrow \frac{2\pi}{3}=120°$

#3 $\Rightarrow \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix}$

$= \begin{pmatrix} z_a \\ x_a \\ y_a \end{pmatrix}$

$\Rightarrow x_a = z_a,\ y_a = x_a,\ z_a = y_a$. Set $x_a = 1$

$\Rightarrow x_a = y_a = z_a = 1$ and the unit vector is $\boxed{\frac{1}{\sqrt{3}}(1,1,1)}$ defines the direction of the axis of rotation

$\Rightarrow$ rotates $\frac{1}{3}$ of way around $\Rightarrow \frac{2\pi}{3} = 120°$

#4 6.3.13

Prove

$$(\vec{A}\times\vec{B})\cdot\left[(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A})\right] = \left[\vec{A}\cdot(\vec{B}\times\vec{C})\right]^2$$

Sol^{n}

let $\vec{D} = \vec{B}\times\vec{C}$

$$\left[\vec{A}\cdot\vec{D}\right]^2 = (\vec{A}\times\vec{B})\cdot\left[\vec{D}\times(\vec{C}\times\vec{A})\right]$$

exchge dot and cross

$$= \left[(\vec{A}\times\vec{B})\times\vec{D}\right]\cdot(\vec{C}\times\vec{A})$$

use BAC-CAB rule after chg'g order of $\times$

$$= \left[-\vec{D}\times(\vec{A}\times\vec{B})\right]\cdot(\vec{C}\times\vec{A})$$

$$= -\left[\vec{A}(\vec{D}\cdot\vec{B}) - \vec{B}(\vec{D}\cdot\vec{A})\right]\cdot\underbrace{(\vec{C}\times\vec{A})}_{\text{vector} \perp \text{ to both } \vec{A} \,\&\, \vec{C}}$$

// to $\vec{A}$

$$= -\left[-\vec{B}(\vec{D}\cdot\vec{A})\right]\cdot(\vec{C}\times\vec{A})$$

(b/c $\vec{A}(\vec{D}\cdot\vec{B}) \perp (\vec{C}\times\vec{A})$)

$(\vec{A}\cdot\vec{D}) = \vec{B}\cdot(\vec{C}\times\vec{A})$; cancelled 1 $(\vec{D}\cdot\vec{A})$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = (\vec{B}\times\vec{C})\cdot\vec{A}$$

exchge dot and cross

~~3~~ 5 6.3.14

Prove $\vec{A}\times(\vec{B}\times\vec{C})+\vec{B}\times(\vec{C}\times\vec{A})+\vec{C}\times(\vec{A}\times\vec{B})$
$= 0$

Sol^{n}

above equals

$$= \left[\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})\right]$$
$$+\left[\vec{C}(\vec{B}\cdot\vec{A})-\vec{A}(\vec{B}\cdot\vec{C})\right]$$
$$+\left[\vec{A}(\vec{C}\cdot\vec{B})-\vec{B}(\vec{C}\cdot\vec{A})\right]$$

$$= -\vec{C}(\vec{A}\cdot\vec{B})+\left[\vec{C}(\vec{B}\cdot\vec{A})-\vec{A}(\vec{B}\cdot\vec{C})\right]+\vec{A}(\vec{C}\cdot\vec{B})$$

$$= -\vec{C}(\vec{A}\cdot\vec{B})+\vec{C}(\vec{B}\cdot\vec{A})$$

$$= 0$$

#~~B~~6.4(6)

$\vec{f} = q\vec{v}\times\vec{B}$ $\quad \perp \vec{v}\ \&\ \vec{B}$

(a) show $\vec{f} \perp \vec{v}$

$$\vec{v}\cdot\vec{f} = q\vec{v}\times(\vec{v}\times\vec{B}) \Rightarrow q\vec{v}\times(\vec{v}\times\vec{B}) = 0$$

(b) show $\vec{v}\cdot\vec{v}$ & $\vec{f}\cdot\vec{f}$ are constants

$$\vec{v}\cdot\vec{f} = m\vec{v}\cdot\frac{d}{dt}\vec{v} = \frac{m}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) = 0$$

$\Rightarrow (\vec{v}\cdot\vec{v})$ is constant (as asked)

c) take $\vec{f}\cdot\vec{f} = (q\vec{v}\times\vec{B})\cdot(q\vec{v}\times\vec{B})$

$$\cancel{= q^2\Big[\vec{v}\big(\vec{B}\cdot\underbrace{[\vec{v}\times\vec{B}]}_{\perp\ \vec{v}\ \&\ \vec{B}}\big) - \vec{B}\big(\vec{v}\cdot\underbrace{[\vec{v}\times\vec{B}]}_{\perp\ \vec{v}\ \&\ \vec{B}}\big)\Big]}$$

$$= (q\vec{v}\times(\vec{B}\times q\vec{v}))\cdot\vec{B}$$

$$= q^2\left[\vec{B}(\vec{v}\cdot\vec{v}) - \vec{v}(\vec{v}\cdot\vec{B})\right]\cdot\vec{B}$$

$$= \underbrace{q^2|\vec{B}|^2|\vec{v}|^2}_{\text{constant}} - q^2|\vec{v}\cdot\vec{B}|^2$$

b/c $\vec{v}$ is in (x-y) plane & $\vec{B}$ is in $\hat{z}$-direction

$\rightarrow |\vec{v}\cdot\vec{B}| = 0$

$\Rightarrow \boxed{\vec{f}\cdot\vec{f} = \text{const}}$

#7 64 (10)

if $\vec{V}(t)$ is a vector function of t, evaluate

$$\int \left(\vec{V} \times \frac{d^2\vec{V}}{dt^2} \right) dt$$

Consider

$$\frac{d}{dt}\left(\vec{V} \times \frac{d\vec{V}}{dt} \right) = \cancel{\frac{d\vec{V}}{dt} \times \frac{d\vec{V}}{dt}}^{\,0} + \vec{V} \times \frac{d^2\vec{V}}{dt^2}$$

$$\Rightarrow \int \left(\vec{V} \times \frac{d^2\vec{V}}{dt^2} \right) dt$$

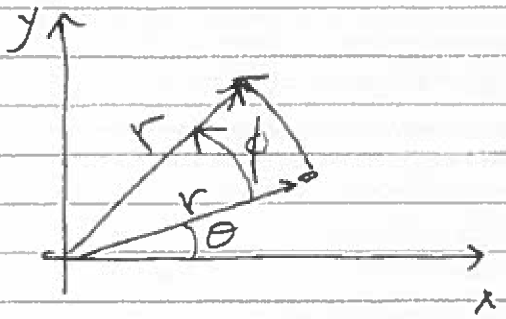
$$= \int \frac{d}{dt}\left[\vec{V} \times \frac{d\vec{V}}{dt} \right] dt$$

$$= \int d \left(\vec{V} \times \frac{d\vec{V}}{dt} \right)$$

constant vector

$$\rightarrow \boxed{\int \left(\vec{V} \times \frac{d^2\vec{V}}{dt^2} \right) dt = \vec{V} \times \frac{d\vec{V}}{dt} + \vec{A}}$$

#8 2∡ cos & sin formulas



rotate $\vec{r}$ by ∡ ϕ about z-axis

$$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{active rotation}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

note: $x' = r\cos(\theta+\phi)$
and
$y' = r\sin(\theta+\phi)$

$$\Rightarrow \left.\begin{aligned} r\cos(\theta+\phi) &= x\cos\phi - y\sin\phi \\ r\sin(\theta+\phi) &= x\sin\phi + y\cos\phi \end{aligned}\right\} (A)$$

note: $$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} (B)$$

$$(A) \,\&\, (B) \Rightarrow \boxed{\begin{aligned} \cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin(\theta+\phi) &= \cos\theta\sin\phi + \sin\theta\cos\phi \end{aligned}}$$